On the Second Order Bifurcation of Limit Cycles from Centers of Planar Analytic Systems

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Abstract

We consider the bifurcation of limit cycles from centers of certain planar polynomial systems of differential equations. Particularly, we apply a new formula that allows a second-order analysis of bifurcations from certain Hamiltonian systems. After making some generalizations about limit cycles produced by perturbing the linear harmonic oscillator, we consider applications of the new formula to some non-linear Hamiltonian systems.

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1 Introduction

The study of limit cycles in planar polynomial systems of differential equations has received much attention since the turn of the century, when in his sixteenth Paris problem David Hilbert asked for a relation between the degree of a polynomial system and the maximum number of limit cycles for that degree. Recent work has focused on limit cycles introduced into systems in bifurcations resulting from various types of perturbations. Particularly, the Melnikov function has provided a reasonably simple technique for finding periodic orbits of systems with centers that are preserved as limit cycles under perturbation. Zeros of the Melnikov function correspond to limit cycles in the perturbed system with a sufficiently small perturbation parameter. According to Peixoto's Theorem [4, p. 301], this preservation of periodic orbits is a type of global bifurcation.

In this paper, we will focus on such global bifurcations of limit cycles from centers of planar polynomial systems. The systems we consider are of the form:

$$\dot{x} = f(x) + \epsilon\, g(x, \epsilon), \tag{1}$$

where $x \in R^2$ and f and g are polynomials in x. In each case, the unperturbed ($\epsilon = 0$) system is Hamiltonian and has a center at the origin, which is surrounded by a continuous band of periodic orbits terminating either at infinity or at a separatrix cycle. We will present the method for a first-order Melnikov analysis of such systems and will then introduce a method for the second-order analysis of systems for which the first-order Melnikov function is identically zero. The latter technique comes from the recent work of I.D. Iliev [3], who provides an equation for the second-order analog of the Melnikov function (M_2).

2 Preliminaries

Before embarking on our applications of Iliev's second-order analysis, we must present some standard definitions and theorems to provide a rough background for the study of bifurcations of limit cycles from a center. We begin by characterizing the basic properties of the kind of system we are studying.

Definition 2.1 *A planar Hamiltonian system is one for which there exists a Hamiltonian function,* $H(x,y) \in C^2(R^2)$*, such that* $\dot{x} = H_y$ *and* $\dot{y} = -H_x$.

Note that a Hamiltonian system is conservative in the sense that $H(x,y)$, or the energy of the system, is constant along any trajectory.

As stated in Section 1, a center of a planar system is surrounded by a *period annulus*, or a continuous band of periodic orbits. A center is structurally unstable, which is to say that a perturbation of a system with a center is no longer topologically equivalent to the unperturbed system. It is because such a perturbation changes the topological structure of the system that we speak of bifurcations from a center as global, as opposed to local bifurcations from hyperbolic critical points or limit cycles.

Definition 2.2 *A limit cycle is an isolated periodic orbit of a system.*

A remarkably useful tool for studying limit cycles is the ***Poincaré map***, or first return map. To construct the Poincaré map, for a system with a periodic orbit we take a hyperplane that intersects that periodic orbit transversally. Then, if we take a point s on the hyperplane within some neighborhood of the intersection of the hyperplane and the periodic orbit, the Poincaré mapping $P(s)$ is obtained by following the orbit through the point s around until it intersects the hyperplane once again. This second intersection is $P(s)$. The function $P(s)$ is continuously differentiable and has a smooth inverse, making it a diffeomorphism. For planar systems, with which we are solely concerned, the hyperplane is merely a line intersecting a periodic orbit transversally, so the Poincaré map is particularly easy to visualize. Note that if $P(s) = s$, then there is a periodic orbit through the point s.

The Poincaré map is useful in deducing the stability of a limit cycle: for instance, if $P(s)$ is closer to the limit cycle than s is, our orbit is spiralling in toward the limit cycle and so it is stable. In fact, the stability of a limit cycle is shown in [4, p. 197] to be determined by the derivative of the Poincaré map. This sort of analysis leads us to define a new function, the ***displacement function*** or $d(s)$, as follows: $d(s) = P(s) - s$. Note that $d'(s) = P'(s) - 1$, so the stability of a limit cycle depends on $d'(s)$. (In fact, the limit cycle is stable if $d'(s) < 0$ and unstable if $d'(s) > 0$.)

In our study we are concerned not so much with the stability of given limit cycles as with the existence of limit cycles in perturbed systems. Because we are perturbing systems with centers, our $P(s)$ and $d(s)$ will depend not only on some distance variable s, but also on our perturbation parameter ϵ. From here out we will consider the displacement function $d(\alpha, \epsilon)$, where α is a monotone function of the distance from the center.

Now, because the displacement function is analytic in ϵ, we may express it as a Taylor series expanded about $\epsilon = 0$. Since $d(\alpha, 0) = 0$, we get:

$$d(\alpha, \epsilon) = \epsilon d_\epsilon(\alpha, 0) + \frac{1}{2}\epsilon^2 d_{\epsilon\epsilon}(\alpha, 0) + \dots$$

By defining new functions $d_1(\alpha) = d_\epsilon(\alpha, 0)$, $d_2(\alpha) = \frac{1}{2}d_{\epsilon\epsilon}(\alpha, 0)$, etc., we can rewrite this series in the following form:

$$d(\alpha, \epsilon) = \epsilon d_1(\alpha) + \epsilon^2 d_2(\alpha) + \dots \tag{2}$$

In our study of global bifurcations from centers, we are interested in finding periodic orbits within the period annulus of the center which are preserved as limit cycles under perturbation. We know that the displacement function is zero for an α value corresponding to a cycle. Thus, finding the isolated zeros of the displacement function for the perturbed system with small $\epsilon > 0$ will tell us which periodic orbits are preserved as limit cycles. Because we have a Taylor series for the displacement function, if we can find isolated zeros of just the first term, $d_1(\alpha)$, then we will know within $O(\epsilon)$ where the limit cycles of the system will lie.

The Melnikov function essentially provides a formula for $d_1(\alpha)$, allowing us to follow the method outlined above for finding limit cycles. A more thorough treatment of the displacement function and its relation to the ϵ-derivative of the Poincaré map may be found in [4, pp. 378-393] or in [1, pp. 343-347], where it is shown that $d_1(\alpha)$ is proportional to $M_1(\alpha)$.

Definition 2.3 *Let $\gamma_\alpha(t)$ be a periodic orbit of the unperturbed system (1) with period T_α and let the wedge product $x \wedge y = x_1 y_2 - y_1 x_2$. Then the Melnikov function for the system (1) is given by*

$$M_1(\alpha) = \int_0^{T_\alpha} f \wedge g(\gamma_\alpha(t), 0)\, dt.$$

We next present the theorem which states the relationship between the zeros of $M_1(\alpha)$ and the limit cycles of system (1). A proof may be found in [1].

Theorem 2.4 *If there exists an isolated, simple zero, α_0, of $M_1(\alpha)$, then for all sufficiently small $\epsilon \neq 0$, the system (1) has a hyperbolic limit cycle in an $O(\epsilon)$ neighborhood of the periodic orbit $\gamma_{\alpha_0}(t)$ of the unperturbed system. Furthermore, if $M_1(\alpha_0) \neq 0$, then for all sufficiently small $\epsilon \neq 0$ system (1) does not have a limit cycle within an $O(\epsilon)$ neighborhood of $\gamma_{\alpha_0}(t)$.*

For many of the examples we will consider, $M_1(\alpha)$ is quite easy to compute. We will provide the results of this first-order analysis when they are relevant to our work. However, we are primarily concerned with cases in which the first-order Melnikov function $M_1(\alpha) \equiv 0$. When this occurs, the lowest-order term in equation (2) will be identically zero, so our first order analysis fails to tell us the zeros of the displacement function. In this case, the lowest-order non-zero term in the series will be the $d_2(\alpha)$ term, so we must find the zeros of this term in order to find the limit cycles of the perturbed system. Until recently, no general formula had been found for $d_2(\alpha)$, so this sort of second-order analysis could only be performed for very specific examples. However, with the recent preprint paper of Iliev [3], we have obtained just such a general formula for polynomial perturbations of certain Hamiltonian systems.

Definition 2.5 *Suppose we have a perturbed planar Hamiltonian system written in the form*

$$\begin{aligned} \dot{x} &= H_y + \epsilon f(x, y, \epsilon), \\ \dot{y} &= -H_x + \epsilon g(x, y, \epsilon), \end{aligned}$$

where f and g are polynomial in x and y and depend analytically on ϵ, and $H(x, y) = \frac{1}{2}y^2 - U(x)$ with deg $U(x) \geq 2$. Then if $M_1(\alpha) \equiv 0$, the second-order Melnikov function may be written

$$\begin{aligned} M_2(\alpha) = &\oint_{\gamma_\alpha(t)} [G_{1\alpha}(x, y)P_2(x, \alpha) - G_1(x, y)P_{2\alpha}(x, \alpha)]\, dx \\ &+ \oint_{\gamma_\alpha(t)} \frac{F(x, y)}{y}[f_x(x, y, 0) + g_y(x, y, 0)]\, dx \\ &+ \oint_{\gamma_\alpha(t)} g_\epsilon(x, y, 0)\, dx - f_\epsilon(x, y, 0)\, dy. \end{aligned}$$

In this formula,

$$F(x,y) = \int_0^y f(x,s,0)\,ds - \int_0^x g(s,0,0)\,ds, \quad G(x,y) = g(x,y,0) + F_x(x,y),$$

$G_1(x,y), G_2(x,y)$ are the odd and even parts of $G(x,y)$ with respect to y,

$$P_2(x,\alpha) = \int_0^x G_2(s, \sqrt{2\alpha + 2U(s)})\,ds, \quad G_{1\alpha}(x,y) = \frac{\partial G_1(x,y)}{\partial y}\left(\frac{1}{y}\right)$$

and α is the energy of the periodic orbit $\gamma_\alpha(t)$.

One can see that this formula is quite involved, so in the work to follow we will not provide the details of the $M_2(\alpha)$ computations, but will merely provide the resulting conclusions. In the work that follows, the variable α is either the energy, h, of the periodic orbit $\gamma_\alpha(t)$ or a quantity that depends monotonically on h. In the latter case, $M_2(\alpha)$ must be multiplied by $\frac{\partial \alpha}{\partial h}$ in order to obtain the displacement function $d_2(\alpha)$.

The following theorem, which is proven in [1], provides the final tool we need to conduct a second-order analysis to find limit cycles.

Theorem 2.6 *The statement of Theorem 2.4 holds true if $M_1(\alpha) \equiv 0$ and if $M_1(\alpha)$ is replaced by $M_2(\alpha)$ in that theorem; i.e. isolated zeros of $M_2(\alpha)$ correspond to limit cycles of (1).*

Iliev states in his paper the expectation that, in general, $M_2(\alpha)$ will have more zeros and will thus produce more limit cycles than $M_1(\alpha)$. This expectation motivates much of our work. We examine systems whose first-order analyses are already known by adjusting their parameters to make $M_1(\alpha) \equiv 0$, making a second-order analysis necessary. We show that in many cases this higher-order analysis can indeed produce more limit cycles than the first-order analysis.

3 Perturbations of the Linear Harmonic Oscillator

In this section we will consider only perturbations of the linear harmonic oscillator, which take the following form:

$$\begin{aligned} \dot{x} &= -y + \epsilon f(x,y,\epsilon), \\ \dot{y} &= x + \epsilon g(x,y,\epsilon). \end{aligned} \tag{3}$$

Chicone and Jacobs have performed a detailed analysis of this system for quadratic perturbation functions f and g. They not only found the $M_1(\alpha)$ function for this generalized system, but went on to perform higher-order analyses all the way up to sixth-order, which they proved produces the maximum possible number of limit cycles. Their results show that one limit cycle can be produced in a second-order analysis, two in a fourth-order analysis, and three in a sixth or any higher degree analysis. For the details of their work see [1, 2].

Because Chicone and Jacobs have already completed this higher-order analysis for the quadratically perturbed linear harmonic oscillator, their formulas provide a check for Iliev's second-order technique. The generalized quadratically perturbed system is written below in Bautin normal form. We shall calculate $M_2(\alpha)$ using Iliev's formula, convert it to $d_2(\alpha)$ and compare it with Chicone and Jacobs' result.

$$\begin{aligned}\dot{x} &= -y + \lambda_1 x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2\end{aligned}$$

with $\lambda_3 = 2\epsilon, \lambda_6 = \epsilon$ and

$$\lambda_i = \sum_{j=1}^{\infty} \lambda_{ij}\epsilon^j$$

for $i = 1, 2, 4, 5$.

Computing Iliev's formula for $M_2(\alpha)$ and then converting it to $d_2(\alpha)$ results in:

$$d_2(\alpha) = 2\pi\lambda_{12}\alpha - \frac{\pi}{4}\lambda_{51}\alpha^3,$$

which agrees with Chicone and Jacobs' result for $d_2(\alpha)$. We performed a numerical calculation for the particular choice of parameters $\lambda_1 = -\frac{1}{2}\epsilon^2$, $\lambda_3 = 2\epsilon$, $\lambda_5 = -4\epsilon$, $\lambda_6 = \epsilon$, $\lambda_4 = -4\epsilon$ and $\lambda_2 = 0$. Plugging the corresponding parameters into the above equation, we see that it predicts a limit cycle at $\alpha = 1$.

The following graph of the displacement function, which plots $\frac{d(\alpha,\epsilon)}{\alpha}$ versus α, shows a zero at the value $\alpha = 1$. Thus, we see that the numerical and analytical results are in excellent agreement. For the graph below, $\epsilon = .0001$

and the vertical scale shows $-.5 \times 10^{-7}$ to $.6 \times 10^{-7}$.

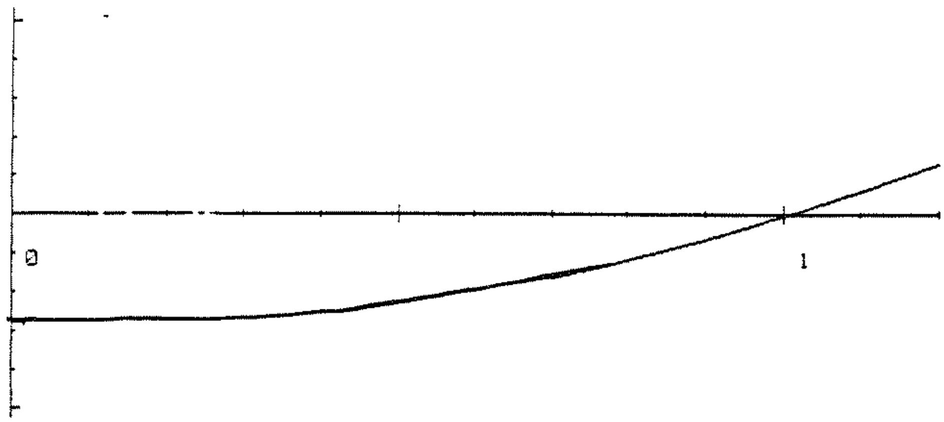

Figure 3.1 – $d(\alpha, \epsilon)/\alpha$ versus α.

While Chicone and Jacobs found more limit cycles by increasing the order of their analysis for a particular system, we are limited for non-quadratic perturbations to a second-order analysis. For this reason, we chose to increase the degree of the perturbation functions f and g in system (3) to determine the impact of the degree on the number of limit cycles generated in a second-order analysis. Our goal was not to study the generalized n-th degree perturbations f and g, but to consider as few terms as possible that will still produce the maximum number of limit cycles for a given degree. We began by setting $g(x, y, \epsilon) = 0$ and considered perturbation functions $f(x, y, \epsilon)$ of varying degrees.

Cubic Perturbations It is known that in a first-order analysis a cubic perturbation of the linear harmonic oscillator can produce one limit cycle. We found that in Iliev's second-order analysis we still produce at most one limit cycle. Because the increase in order of analysis does not affect the number of limit cycles for the cubic case, we omit the details of our calculations and move on to more interesting cases.

4th-Degree Perturbations This is the first higher-degree perturbation which produces interesting results. After some trial and error and eventual generalization of how different terms in the perturbation functions affect

$M_2(\alpha)$, we settled on the following system as the simplest case introducing two limit cycles under second-order analysis. Note that $g(x, y, \epsilon) = 0$ in our system, and we have changed the sign of y from system (3) so that this system meets the conditions necessary to apply Iliev's formula from Definition 2.5:

$$\begin{aligned}
\dot{x} &= y + \epsilon(a(\epsilon)x - d(\epsilon)x^3 - bxy + ex^4), \\
\dot{y} &= -x,
\end{aligned}$$

in which $a(\epsilon)$ and $d(\epsilon)$ are power series in ϵ. If $a_0, d_0 \neq 0$, then a first-order Melnikov analysis results in $M_1(\alpha) = \pi\alpha^2(-a_0 + \frac{3}{4}d_0\alpha^2)$, which has one positive zero implying one limit cycle. However, if $a_0 = d_0 = 0$, then $M_1(\alpha) \equiv 0$ and

$$M_2(\alpha) = \pi\alpha^2 \left(\frac{-5}{24} be\alpha^4 - \frac{3}{4} d_1\alpha^2 + a_1 \right),$$

which has as many as two positive zeros with properly chosen a_1, b, d_1, and e. If any one of these four parameters were 0, we would get less than two limit cycles. Thus, we can see this is the simplest quartic perturbation for which an increase to second-order analysis increases the number of limit cycles. As we will soon see, a pattern emerges for higher even-degree perturbations in the sorts of terms required to produce additional limit cycles.

5th-Degree Perturbations It is known that in a first-order analysis a 5th-degree perturbation can produce as many as two limit cycles. As with the cubic case, we found that a second-order analysis does not increase the number of limit cycles of the system if $f(x, y, \epsilon)$ is of degree 5 and $g(x, y, \epsilon) = 0$. Thus, once again, we will omit the details for this case.

Furthermore, we note that it is possible in first-order analysis to produce n limit cycles with a $(2n+1)$st degree perturbation [1, p. 348]. We noticed that, due to the nature of Iliev's formula, $(2n+1)$st degree perturbations f cannot produce more than n limit cycles in second-order analysis as well. Essentially, this is because Iliev's formula will never produce an $M_2(\alpha)$ with a high enough power of α in any term to conceivably have $n+1$ zeros. For this reason, we will consider our second-order analysis of odd-degree perturbations f, when g is zero, to be complete.

6th-Degree Perturbations The following system is the simplest one for which we found three limit cycles under second-order analysis. It follows closely the pattern established by our 4th-degree perturbation, with the addition of an x^5 term and the replacement of the x^4 term by an x^6 term:

$$\begin{aligned}\dot{x} &= y + \epsilon(a(\epsilon)x - d(\epsilon)x^3 + f(\epsilon)x^5 - bxy + gx^6),\\ \dot{y} &= -x.\end{aligned}$$

Once again, a, d and f are power series in ϵ. The first-order analysis results in $M_1(\alpha) = \pi\alpha^2(-a_0 + \frac{3}{4}d_0\alpha^2 - \frac{5}{8}f_0\alpha^4)$, which has at most two positive zeros. If $a_0 = d_0 = f_0 = 0$, then again $M_1(\alpha) \equiv 0$ so we move to the second-order analysis and find

$$M_2(\alpha) = \pi\alpha^2\left(\frac{-35}{64}bg\alpha^6 + \frac{5}{8}f_1\alpha^4 - \frac{3}{4}d_1\alpha^2 + a_1\right),$$

which has as many as three positive zeros if the parameters are properly chosen. Again, if any of the five parameters a_1, b, d_1, f_1 and g were zero then $M_2(\alpha)$ could no longer have three positive zeros. Thus, we have established a basic pattern for the simplest even-degree perturbations which result in the maximal number of limit cycles under second-order analysis.

Higher Degree Perturbations We have already stated that for odd perturbations f of degree $2n+1$, both first- and second-order analyses will produce at most n limit cycles. We have also established a pattern for even perturbations of degree $2n$. As long as the function f includes each odd power of x less than $2n$, an xy term and an x^{2n} term, the structure of Iliev's formula guarantees the following statement.

Observation 3.1 *For system (3) with $g(x,y,\epsilon) = 0$ and $f(x,y,\epsilon)$ of degree $2n$, first-order analysis yields at most $n-1$ limit cycles and second-order analysis yields at most n limit cycles.*

Up to this point, we have only considered perturbations with $g(x,y,\epsilon) = 0$. If we allow $g(x,y,\epsilon) \neq 0$ and take f and g both of degree n, the second-order analysis actually shows we can have even more limit cycles than in the previous discussion. We first consider an example with $deg(f) = deg(g) = 4$.

Example: Consider the following quartically perturbed harmonic oscillator:

$$\begin{aligned}\dot{x} &= y + \epsilon(\epsilon ax + bxy + \epsilon dx^3 + ex^4),\\ \dot{y} &= -x + \epsilon cx^4.\end{aligned}$$

Note that in this system, $f(x,y,\epsilon) = \epsilon ax + bxy + \epsilon dx^3 + ex^4$ and $g(x,y,\epsilon) = cx^4$. For this system, $M_1(\alpha) \equiv 0$, so we proceed to a second-order analysis and, using Iliev's formula, we find

$$M_2(\alpha) = \pi\alpha^2 \left(\frac{7}{16}c\alpha^6 + \frac{5}{24}be\alpha^4 + \frac{3}{4}d\alpha^2 + a \right).$$

If the non-zero parameters a, b, c, d and e are chosen properly, $M_2(\alpha)$ will have as many as three positive zeros. So, we see that this system can have as many as three limit cycles, and it does in fact have exactly three limit cycles for an appropriate choice of parameters.

Similarly, it can be shown that a fifth-degree perturbation (where $deg(f) = deg(g) = 5$) can also produce three limit cycles in second-order analysis. By inspecting the terms that arise in Iliev's formula for even perturbations f and g, we can summarize the trend in the following observation.

Observation 3.2 *For system (3) with $f(x,y,\epsilon)$ and $g(x,y,\epsilon)$ of degree $2n$, a first-order analysis yields at most $n-1$ limit cycles and a second-order analysis yields at most $2n-1$ limit cycles. There are perturbations, f and g, of degree $2n$ which produce $2n-1$ limit cycles.*

Table 3.1 below summarizes the results of this section. It shows the basic patterns we have noticed for the number of limit cycles possible under first- and second-order analysis for different degree polynomial perturbations.

Perturbation Degree	k=1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	1	1	2	2	3
3	1	1				
4	1	3				
5	2	3				
⋮	⋮	⋮				
2m	m-1	2m-1				
2m+1	m	2m-1				

Table 3.1 - The maximum number of limit cycles obtainable from a kth-order analysis of the harmonic oscillator with increasing degree polynomial perturbations.

The second row of the table reflects the work of Chicone and Jacobs [2], which provides a complete analysis of the number of limit cycles possible for any quadratic perturbation of the linear harmonic oscillator. An interesting problem remains in extending the rows of this table for higher degree perturbations.

4 Perturbations of a Cubic System

We now turn our attention to what was the ultimate goal of our research: to apply Iliev's method to a more complicated nonlinear Hamiltonian sytem with a center. We selected as an example a system for which a first-order analysis has already been performed. The perturbed truncated pendulum system, which is treated in [1, pp. 362-5], is as follows:

$$\begin{aligned} \dot{x} &= y + \epsilon h(x, y, \epsilon), \\ \dot{y} &= -x + x^3. \end{aligned} \tag{4}$$

The first-order analysis described in [1] is done for the particular perturbation function $h(x, y, \epsilon) = ax + x^3$. It produces at most one limit cycle. Because the equations of the periodic orbits within the period annulus of this system are significantly more complex than those of the linear harmonic oscillator, even the first-order analysis for this system is fairly complicated. It involves integrating even powers of the Jacobi elliptic functions, namely $sn^2 u$, $sn^4 u$ and $sn^6 u$. The second-order analysis involves exactly these same integrals.

The analysis can be completed for choices of h similar to the one above. For instance, if $h(x, y, \epsilon) = \epsilon ax + \epsilon cx^3$, then the second-order analysis results in one limit cycle, the same as the first-order analysis. But this adds nothing new, since if an ϵ is factored out of $h(x, y, \epsilon)$ we see that the same first-order analysis as before can be performed, replacing ϵ with ϵ^2. We want to consider a perturbation function that has the possibility of producing more zeros in its second-order analysis than in the first-order analysis. We selected the simplest possible system of this type:

$$\begin{aligned} \dot{x} &= y + \epsilon(\epsilon ax + bx^2 + cxy), \\ \dot{y} &= -x + x^3. \end{aligned}$$

The perturbation $h(x, y, \epsilon)$ has been chosen here so that $M_1(\alpha) \equiv 0$, so we proceed to the second-order analysis. The calculation of $M_2(\alpha)$ requires taking integrals of the elliptic functions sn^2u, sn^4u and sn^6u, much as in [1, pp. 362-3]. The resulting expression is given in terms of $K(\alpha)$ and $E(\alpha)$, the complete elliptic integrals of the first and second kind, respectively:

$$M_2(\alpha) = -48bc\left(1+\alpha^2\right)^{\frac{-5}{2}}\alpha^2\left[K(\alpha)-E(\alpha)\right]$$

$$+\frac{176}{9}bc\left(1+\alpha^2\right)^{\frac{-3}{2}}\left[\left(2+\alpha^2\right)K(\alpha)-2\left(1+\alpha^2\right)E(\alpha)\right]$$

$$-\frac{304}{75}bc\left(1+\alpha^2\right)^{\frac{-5}{2}}\left[\left(4\alpha^4+3\alpha^2+8\right)K(\alpha)-\left(8\alpha^4+7\alpha^2+8\right)E(\alpha)\right]$$

$$+8a\left(1+\alpha^2\right)^{\frac{-1}{2}}\left[K(\alpha)-E(\alpha)\right]$$

$$-\frac{16}{3}a\left(1+\alpha^2\right)^{\frac{-3}{2}}\left[\left(2+\alpha^2\right)K(\alpha)-2\left(1+\alpha^2\right)E(\alpha)\right].$$

Finding the positive zeros of this function is a bit trickier than in the earlier examples. Simplification shows that $M_2(\alpha) = 0$ when

$$\frac{a}{bc} = \frac{-3\left[\left(94\alpha^4-42\alpha^2+188\right)K(\alpha)-\left(188\alpha^4+52\alpha^2+188\right)E(\alpha)\right]}{225\left(1+\alpha^2\right)\left[\left(\alpha^2-1\right)K(\alpha)+\left(\alpha^2+1\right)E(\alpha)\right]}.$$

It turns out that we can have two zeros for appropriate choices of a, b and c. In fact, if $b = c = 1$, we see from numerical work that we get two limit cycles for $a = .01$. See Figure 4.1 below, which plots $d(x,\epsilon)/x$ versus x for $0 < x < 1$ where $\alpha^2 = x^2/(2-x^2)$, $b = c = 1$, $\epsilon = .01$ and a has the three values $a = .06$, .01 and -.01. It is clear that x, and therefore α, has two positive zeros for $a = .01$, so we get two limit cycles in this case.

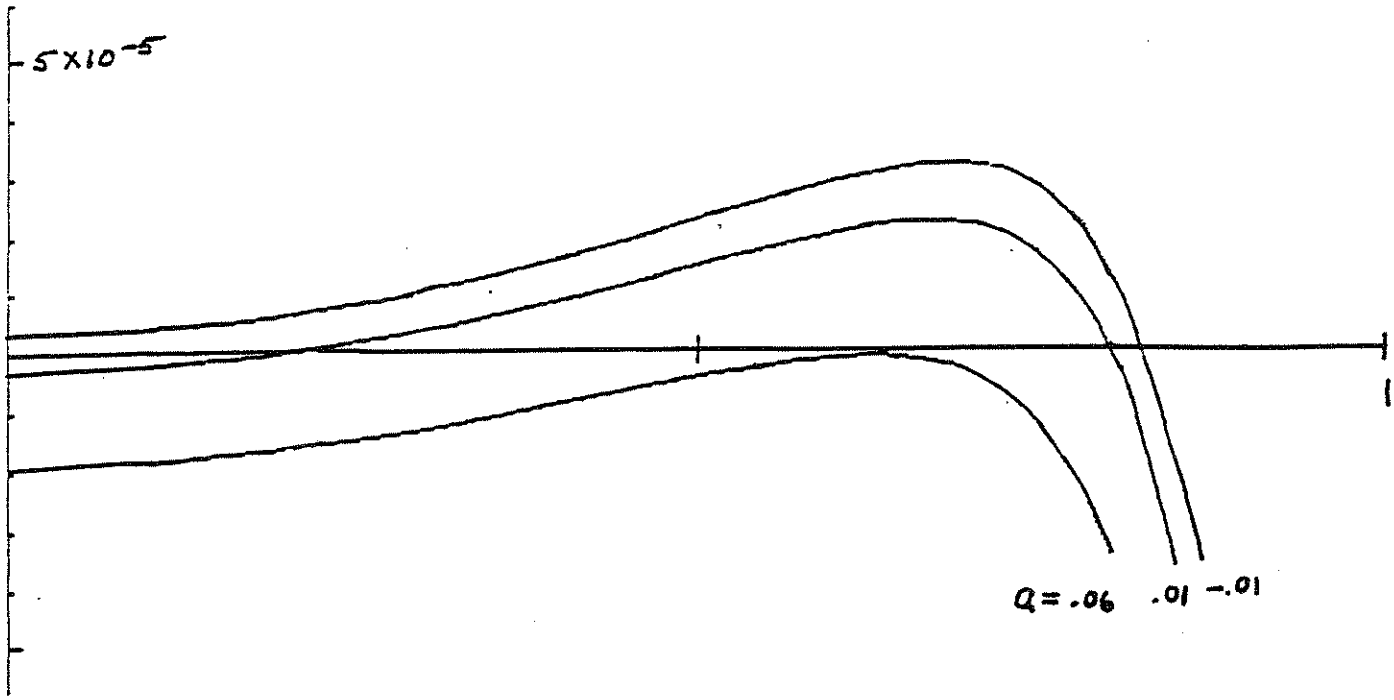

Figure 4.1

Hence, for a quadratically perturbed cubic system, we see that two limit cycles are possible in second-order analysis. Thus, it is clear that this cubic system is capable of producing more limit cycles than the linear harmonic oscillator under polynomial perturbation. Much work remains to be done in order to complete a chart analogous to Table 3.1 for this and other other similar nonlinear systems.

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