Using Symmetry Preserving Maps to Determine the Symmetries of Attractors for Dynamical Systems

A Summary of Work done for a NSF REU program at Northern Arizona University *

Jeffrey H. Schenker
University of Wyoming
Laramie, Wy 82071
deroble@plains.uwyo.edu

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Abstract

The concept of symmetry in dynamical systems is considered. Several lemmas regarding isotropy subgroups and isotropy equivalence are presented. A theorem regarding the density of symmetry preserving maps in $C^k(\mathbf{M}, \mathbf{V})$ where $\mathbf{M}$ and $\mathbf{V}$ are real vector spaces is proved. Potential applications to differential equations with $\mathbf{D}_n$ and $\mathbf{D}_n \times \mathbf{Z}_2$ symmetry are indicated.

1 Introduction

The phase space of many dynamical systems is of high enough dimension so as to make it difficult to visualize the attractors (See, for example, [7] in which the phase space is $\mathbf{R}^{12}$). We wish to visualize the symmetries of attractors by mapping them into low dimensional spaces (preferably dim ≤ 3) in such a way that symmetry is preserved.

Recently, a theory of *symmetry detectives* has been developed [2, 4, 1]. Symmetry detective theory provides a method for *computing* the symmetry of an attractor for a dynamical system. The work presented herein proposes a method in which the symmetry of an attractor is *observed visually.*

*Advisor: Dr. James W. Swift, Northern Arizona University

1.1 Dynamical Systems

There are two ways to construct a dynamical system from a map:

$$f : \mathbf{R}^n \to \mathbf{R}^n \tag{1.1}$$

1. The function can be used to generate a discrete system in which the orbits:

$$x_0 \overset{f}{\mapsto} x_1 \overset{f}{\mapsto} x_2 \overset{f}{\mapsto} x_3 \overset{f}{\mapsto} \cdots \tag{1.2}$$

   are considered for various starting values $x_0 \in \mathbf{R}^n$.

2. The function can be used to generate the vector field of a first order ODE:

$$\dot{x} = f(x) \tag{1.3}$$

   In this case, the orbits are the solution curves to the ODE.

If f defines a dynamical system in either of these ways, we will say that the dynamical system *stems from* f.

For either a discrete system or an ODE, *attractors* are periodic, quasi-periodic, or chaotic orbits to which nearby orbits "tend." There is no universally accepted definition of an attractor, but the following properties are fairly general:

1. An attractor contains a dense topologically transitive orbit.

2. An attractor is contained in an open basin of attraction.

1.2 Symmetries and Dynamical Systems

Suppose that there is a dynamical system which stems from $f : \mathbf{R}^n \to \mathbf{R}^n$. Suppose that some finite group, Γ, acts faithfully on $\mathbf{R}^n$ (for a discussion of group actions see [6, ch. 4]). If

$$f(\gamma \bullet x) = \gamma \bullet f(x) \quad \forall \gamma \in \Gamma \tag{1.4}$$

then f is said to be Γ *equivariant.*

Lemma 1.1 *If A is an attractor with an open basin of attraction for a dynamical system which stems from a Γ equivariant function then:*

$$\gamma A = A \text{ or } \gamma A \cap A = \emptyset \quad \forall \gamma \in \Gamma. \tag{1.5}$$

Lemma 1.1 is proved in [3].

We define the symmetry subgroup of an attractor (or an arbitrary set) to be:

$$\Sigma(A) := \{\gamma \in \Gamma \mid \gamma A = A\}. \tag{1.6}$$

If A is an attractor for a Γ equivariant function, it may not be true that $\Sigma(A) = \Gamma$. Occurences of this fact can be found even in low dimensional systems such as the Lorenz Equations:

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned} \qquad (1.7)$$

These equations are equivarient under the transformation $(x, y, z) \mapsto (-x, -y, z)$, which can be considered an action of $\mathbf{Z}_2$ on $\mathbf{R}^3$. Numerical simulations indicate that at $\sigma = 10,\ \rho = 93,\ \beta = \frac{8}{3}$ there is an attracting periodic orbit which has full symmetry ($\mathbf{Z}_2$). At $\sigma = 10,\ \rho = 100,\ \beta = \frac{8}{3}$ there is an attracting periodic orbit which has trivial symmetry ($\{1\}$).

Nevertheless, if A is an attractor then, given any $\gamma \in \Gamma$, γA is an attractor. If $\gamma \in \Sigma(A)$ then $\gamma A = A$, otherwise γA is said to be *conjugate* to A.

1.3 Group Representations and Isotropy

If the action of Γ on a vector space V is linear, we say that V is a *representation* of Γ (more strictly, the group homomorphism $\Gamma \to \mathrm{GL}(V)$ induced by the action is a representation and V is a representation space).

It is possible to define the symmetry group of a single point $x \in V$:

$$\Sigma(x) := \Sigma(\{x\}) = \{\gamma \in \Gamma \mid \gamma x = x\} \qquad (1.8)$$

If $G \leq \Gamma$ and $G = \Sigma(x)$ for some $x \in V$ then G is said to be an *isotropy subgroup* for the representation V. For a set $A \subseteq V$ we define:

$$T(A) := \{\gamma \in \Gamma \mid \gamma x = x\ \forall x \in A\} \qquad (1.9)$$

$T(A)$ is called the isotropy subgroup of A. Note that:

$$T(A) = \bigcap_{x \in A} \Sigma(x). \qquad (1.10)$$

In Lemma 2.3 we show that the intersection of two isotropy subgroups is an isotropy subgroup. Since we consider only finite groups, the intersection in Equation (1.10) is finite. Thus $T(A)$ is an isotropy subgroup.

We define the fixed point subspace of a subgroup:

$$\mathrm{Fix}_V(H) = \{x \in V \mid hx = x\ \forall h \in H\} \qquad (1.11)$$

where $H \leq \Gamma$. Note that:

1. $\mathrm{Fix}_V(H)$ is a vector space.

2. If $H \leq G$ then $\mathrm{Fix}_V(G) \subseteq \mathrm{Fix}_V(H)$.

2 Isotropy Equivalence

The work on symmetry preserving maps contained herein and the work on symmetry detectives in [1] depend on the notion of *isotropy equivalent* representations. In this section, we develop some theory regarding this notion.

Definition 2.1 *We say that two representations of a finite group Γ, V_1 and V_2 are* isotropy equivalent *if H is an isotropy subgroup for the action of Γ on V_1 if and only if H is an isotropy subgroup for the action of Γ on V_2*

We will use the notation $V_1 \sim V_2$ to indicate isotropy equivalence when the group Γ is understood. Note that $\sim$ is an equivalence relation.

The following three lemmas develop several properties of isotropy subgroups.

Lemma 2.1 *Let Γ be a finite group with representation V. Let $H \leq \Gamma$. The following are equivalent:*

1. *H is an isotropy subgroup.*
2. *For all $G \leq \Gamma$ such that $G \geq H$, $Fix_V(G) = Fix_V(H)$ if and only if G=H.*
3. *$Fix_V(G) \subset Fix_V(H)$ for all $G \leq \Gamma$ such that $G > H$.*

We take $\subset$ and $<$ to mean proper subset and proper subgroup respectively.

Proof:

(1 $\Rightarrow$ 2) Assume that H is an isotropy subgroup. Let $G \leq \Gamma$ such that $G \geq H$. Clearly if $G = H$ then $\mathrm{Fix}_V(G) = \mathrm{Fix}_V(H)$. If $\mathrm{Fix}_V(G) = \mathrm{Fix}_V(H)$ then there exists $v \in \mathrm{Fix}_V(H)$ such that $H = \Sigma(v)$ (since H is an isotropy subgroup). But then $v \in \mathrm{Fix}_V(G)$ and since $H = \Sigma(v)$, $G \leq H$. Thus $G = H$.

(2 $\Rightarrow$ 3) Assume that or all $G \leq \Gamma$ such that $G \geq H$, $\mathrm{Fix}_V(G) = \mathrm{Fix}_V(H)$ if and only if G=H. Let $G \leq \Gamma$ such that $G > H$. Then $\mathrm{Fix}_V(G) \neq \mathrm{Fix}_V(H)$. But, $\mathrm{Fix}_V(G) \subseteq \mathrm{Fix}_V(H)$, so $\mathrm{Fix}_V(G) \subset \mathrm{Fix}_V(H)$.

(3 $\Rightarrow$ 1) Assume that $\mathrm{Fix}_V(G) \subset \mathrm{Fix}_V(H)$ for all $G \leq \Gamma$ such that $G > H$. Let $\mathcal{G} = \{G \leq \Gamma \mid H < G\}$. Now $\mathrm{Fix}_V(G) \subset \mathrm{Fix}_V(H)$ for all $G \in \mathcal{G}$. But each of the fixed point subspaces are *subspaces*, consequently, there exists

$$v \in \mathrm{Fix}_V(H) \setminus \bigcup_{G \in \mathcal{G}} \mathrm{Fix}_V(G). \tag{2.1}$$

But this implies that $H = \Sigma(v)$, so H is an isotropy subgroup.

□

Lemma 2.2 *Let Γ be a finite group with representation V. If $H \leq \Gamma$ then there exists $F \leq \Gamma$ such that $H \leq F$, $Fix_V(H) = Fix_V(F)$, and F is an isotropy subgroup.*

Proof: Let F be the subgroup of pointwise symmetries of $\mathrm{Fix}_V(H)$, *i.e.*:

$$F = \{\gamma \in \Gamma \mid \gamma v = v \;\; \forall v \in \mathrm{Fix}_V(H)\}. \tag{2.2}$$

Clearly, $H \leq F$ and $\mathrm{Fix}_V(F) = \mathrm{Fix}_V(H)$. Now let $J \leq \Gamma$ such that $J > F$. Then, there exists $j \in J$ and $v \in \mathrm{Fix}_V(H)$ such that $jv \neq v$. Thus, $\mathrm{Fix}_V(J) \subset \mathrm{Fix}_V(F)$. Thus, by Lemma 2.1, F is an isotropy subgroup. □

Lemma 2.3 *Let $H, G \leq \Gamma$ be isotropy subgroups for a representation V. Then $H \cap G$ is an isotropy subgroup.*

Proof: If $H = G$ this is trivial, so assume $H \neq G$. Clearly $\mathrm{Fix}_V(G) \subseteq \mathrm{Fix}_V(G \cap H)$ and $\mathrm{Fix}_V(H) \subseteq \mathrm{Fix}_V(G \cap H)$. Let $F \leq \Gamma$ such that $F > G \cap H$. Then there is $f \in F$ such that $f \notin G$ or $f \notin H$. Since H and G are isotropy subgroups, either $\mathrm{Fix}_V(G) \not\subseteq \mathrm{Fix}_V(F)$ or $\mathrm{Fix}_V(H) \not\subseteq \mathrm{Fix}_V(F)$. Thus $\mathrm{Fix}_V(F) \neq \mathrm{Fix}_V(G \cap H)$. Thus by Lemma 2.1, $H \cap G$ is an isotropy subgroup. □

Note that Γ is always an isotropy subgroup. If the action of Γ on V is faithful, then $\mathbf{1}$ is an isotropy subgroup, otherwise the kernel of the action is an isotropy subgroup. In fact, Lemma 2.2 indicates that every subgroup is contained in a "smallest" isotropy subgroup.

Now we present two lemmas. The first of these provides a method for showing that two representations are isotropy equivalent. The second allows us to reduce the size of representations by removing isotropically equivalent representations which are "redundant."

Lemma 2.4 *Let Γ be a finite group with representations V_1 and V_2. Suppose that:*

$$Fix_{V_1}(G) = Fix_{V_1}(H) \Longleftrightarrow Fix_{V_2}(G) = Fix_{V_2}(H) \tag{2.3}$$

whenever $H \leq G \leq \Gamma$. Then $V_1 \sim V_2$.

Proof: H is an isotropy subgroup for V_1 if and only if for every $G \leq \Gamma$ such that $G \geq H$, $G = H$ whenever $\mathrm{Fix}_{V_1}(G) = \mathrm{Fix}_{V_1}(H)$ (see Lemma 2.1). Thus by hypothesis, H is an isotropy subgroup for V_1 if and only if for every $G \leq \Gamma$ such that $G \geq H$, $\mathrm{Fix}_{V_2}(G) = \mathrm{Fix}_{V_2}(H)$ whenever $G = H$. Therefore by Lemma 2.1 H is an isotropy subgroup for V_1 if and only if H is an isotropy subgroup for V_2. Thus $V_1 \sim V_2$. □

Lemma 2.5 *Let Γ be a finite group with representations $V_\perp$, V_1, and V_2. If $V_1 \sim V_2$ then $V_\perp \oplus V_1 \oplus V_2 \sim V_\perp \oplus V_1$.*

Proof: Clearly, if H is an isotropy subgroup for $V_\perp \oplus V_1$ then H is an isotropy subgroup for $V_\perp \oplus V_1 \oplus V_2$. Now let H be an isotropy subgroup for $V_\perp \oplus V_1 \oplus V_2$. We claim that H=$F \cap G_1 \cap G_2$ for some isotropy subgroups F for $V_\perp$, G_1 for V_1 and G_2 for V_2. To show this, note that if $H = \Sigma((v_\perp, v_1, v_2))$ then elements of H must fix $(v_\perp, v_1, v_2)$ coordinate-wise, thus $H \leq F \cap G_1 \cap G_2$ where $F = \Sigma(v_\perp)$, $G_1 = \Sigma(v_1)$, $G_2 = \Sigma(v_2)$. However, any element of $F \cap G_1 \cap G_2$ fixes $(v_\perp, v_1, v_2)$, so $F \cap G_1 \cap G_2 = H$. By Lemma 2.3, $G_1 \cap G_2$ is an isotropy subgroup of V_1. Therefore H is an isotropy subgroup for $V_\perp \oplus V_1$. □

3 Symmetry Preserving Maps

3.1 Some Definitions

Let $\mathbf{M}$ and $\mathbf{V}$ be real representations of a finite group Γ. Suppose that the action of Γ on $\mathbf{M}$ is faithful. Let a dynamical system stem from a Γ equivariant $f : \mathbf{M} \to \mathbf{M}$. Let

$$\mathcal{A}(\mathbf{M}) = \{A \subseteq \mathbf{M} \mid A \text{ is compact and } \gamma A \cap A = \emptyset \text{ or } \gamma A = A \ \ \forall \gamma \in \Gamma\} \quad (3.1)$$

Then, by Lemma 1.1, all bounded attractors for the dynamical system are elements of $\mathcal{A}(\mathbf{M})$.

Let $C^k_\Gamma(\mathbf{M}, \mathbf{V})$ be the set of all k times differentiable Γ equivariant functions from $\mathbf{M}$ to $\mathbf{V}$. This is a topological space under the standard C^k distance (see [5]).

Lemma 3.1 *For any $\phi \in C^k_\Gamma(M, V)$, $A \subset M$:*

1. $\Sigma(A) \leq \Sigma(\phi(A))$
2. $T(A) \leq T(\phi(A))$

Proof: Let $\rho \in \Sigma(A)$. Then $\rho\phi(A) = \phi(\rho A) = \phi(A)$. Thus $\rho \in \Sigma(\phi(A))$. The remaining item is similar. □

3.2 A Theorem Regarding Symmetry Preserving Maps

Note that $\mathbf{V} = \mathbf{R}^m$ for some m. We consider $S^{m-1} \subset \mathbf{V}$. We can represent all elements $v \in V$, $v \neq 0$ uniquely as a product:

$$v = r\hat{s} \quad (3.2)$$

for some $r \in [0, \infty)$ and $\hat{s} \in S^{m-1}$. Because we consider $S^{m-1} \subset \mathbf{V}$, Γ can act on elements of S^{m-1}. We assume that the action is orthogonal so that $\gamma\hat{s} \in S^{m-1}$ for all $\gamma \in \Gamma$ and $\hat{s} \in S$. Thus, given $r\hat{s} \in \mathbf{V}$, $\gamma(r\hat{s}) = r(\gamma\hat{s})$ for all $\gamma \in \Gamma$.

Suppose we have a compact set $U \subset \mathbf{V}$. We define $X_U : S^{m-1} \to [0, \infty)$:

$$X_U(\hat{s}) := \begin{cases} r & \text{maximal such that } r\hat{s} \in U \\ 0 & \text{if } r\hat{s} \notin U \text{ for any } r \in [0, \infty) \end{cases} \tag{3.3}$$

The compactness of U guarantees that it is closed and bounded, so this function is well defined at every point.

Theorem 3.1 *Let Γ act faithfully on $\boldsymbol{M}$ and orthogonally on $\boldsymbol{V}$. Let $\mathcal{A}(\boldsymbol{M})$ be defined as above. Let $A \in \mathcal{A}(\boldsymbol{M})$. If $T(A)$ is an isotropy subgroup for $\boldsymbol{V}$, then:*

1. *There exists an open dense set $\mathcal{O} \subset C^k_\Gamma(\boldsymbol{M}, \boldsymbol{V})$ such that $T(\phi(A)) = T(A)$ for all $\phi \in \mathcal{O}$.*
2. *If $\phi \in C^k_\Gamma(\boldsymbol{M}, \boldsymbol{V})$ such that there is some point $a \in A$ for which $\phi(a)$ has isotropy $T(A)$ then there exists $\tilde{\phi} \in C^k_\Gamma(\boldsymbol{M}, \boldsymbol{V})$ arbitrarily close to ϕ (with respect to the C^k topology) such that:*

$$\Sigma(A) = \Sigma(\tilde{\phi}(A)) \tag{3.4}$$

Proof: Suppose that $\psi \in C^k_\Gamma$ such that $T(\psi(A)) \geq T(A)$. Pick a point $a \in A$ such that $\Sigma(a) = T(A)$. Pick $v \in \mathbf{V}$ such that $\Sigma(v) = T(A)$.

Let $\delta > 0$ such that $B_\delta(a) \cap \gamma B_\delta(a) = \emptyset$ for all $\gamma \in \Gamma$, $\gamma \notin T(A)$. Let $\eta : \mathbf{M} \to \mathbf{R}$ have support $B_\delta(a)$ with $\eta(a) = 1$.

Let $\epsilon > 0$. Define:

$$\phi(x) = \psi(x) + \epsilon \sum_{\gamma \in \Gamma} \gamma v \eta(\gamma^{-1} x) \tag{3.5}$$

Now $\phi(a) = \psi(a) + \epsilon |T(A)| v$. For ϵ small enough, $\Sigma(\phi(a)) = T(A)$. Thus, $T(\phi(a)) \leq T(A)$. But ϕ is Γ equivarient, so $T(\phi(a)) \geq T(A)$. Since ϵ may be as small as we like, $\mathcal{O}$ is dense.

Openess is clear since points with isotropy $T(A)$ are dense in $\mathrm{Fix}_{\mathbf{V}}(T(A))$. This concludes the proof of item 1.

Define

$$R = \max_{x \in A} |\phi(x)| = \max_{\hat{s} \in S^{m-1}} X_{\phi(A)}(\hat{s}) \tag{3.6}$$

Pick $\beta \in \Gamma$ such that

$$X_{\phi(A)}\left(\beta \frac{\phi(a)}{|\phi(a)|}\right) = \max_{\gamma \in \Gamma} X_{\phi(A)}\left(\gamma \frac{\phi(a)}{|\phi(a)|}\right) \tag{3.7}$$

Let $\hat{v} = \beta \frac{\phi(a)}{|\phi(a)|}$. Let $b \in A$ such that $\phi(b) = X_{\phi(A)}(\hat{v})\hat{v}$.

Note that $\phi(A) \subseteq B_R(0)$ and $\phi(b) \leq R$. We wish to define an invertible Γ equivarient function $\psi : \mathbf{V} \to \mathbf{V}$ such that

$$\begin{array}{ll} 1) & \psi \circ \phi(A) \subseteq B_R(0) \\ 2) & \psi \circ \phi(b) = R \end{array} \tag{3.8}$$

We wish ψ to be invertible and Γ equivarient so that:

$$\Sigma(\psi(U)) = \Sigma(U) \tag{3.9}$$

for any set $U \subseteq \mathbf{V}$.

Define $H : S^{m-1} \to [0, \infty)$ with the following properties:

$$\begin{array}{ll} 1) & H(\hat{v}) = \frac{R}{|\phi(b)|} - 1. \\ 2) & H(\hat{s}) = 0 \ \forall \hat{s} \in S^{m-1}, \ \hat{s} \neq \hat{v}. \end{array} \tag{3.10}$$

We define a Γ invariant extension of H:

$$\hat{H}(\hat{s}) = \frac{1}{|\Sigma(\hat{v})|} \sum_{\gamma \in \Gamma} H(\gamma \hat{s}). \tag{3.11}$$

$\hat{H}$ satisfies two properties, similar to those of H:

$$\begin{array}{ll} 1) & \hat{H}(\gamma \hat{v}) = \frac{R}{|\phi(b)|} - 1 \ \forall \gamma \in \Gamma. \\ 2) & \hat{H}(\hat{s}) = 0 \ \forall \hat{s} \in S^{m-1}, \ \hat{s} \neq \gamma \hat{v} \text{ for some } \gamma \in \Gamma. \end{array} \tag{3.12}$$

Note that for any $\gamma \in \Gamma$, $|\Sigma(\hat{v})| = |\Sigma(\gamma \hat{v})| = |T(A)|$. Consequently, by the choice of β:

$$(1 + \hat{H}(\hat{s})) X_{\phi(A)}(\hat{s}) \leq R \ \forall \hat{s} \in S^{m-1} \tag{3.13}$$

We can use $\hat{H}$ to define an invertible function, $\psi : \mathbf{V} \to \mathbf{V}$:

$$\psi(r\hat{s}) = \left[r(1 + \hat{H}(\hat{s})) \right] \hat{s} \tag{3.14}$$

It is easy to check that ψ is Γ equivarient and satisfies the properties in Equation (3.8).

Let $\delta > 0$ such that $B_\delta(\rho b) \cap A = \emptyset$ whenever $\rho \notin \Sigma(A)$ and $B_\delta(\rho b) \cap B_\delta(b) = \emptyset$ whenever $\rho \notin T(A)$. Let $\eta \in C^k(\mathbf{M}, \mathbf{R})$ have support $B_\delta(b)$ with $\eta(b) = 1$.

Let $\epsilon > 0$. Define $\tilde{\phi}$:

$$\psi \circ \tilde{\phi}(x) = \left(1 + \epsilon \sum_{\gamma \in \Gamma} \eta(\gamma x) \right) \psi \circ \phi(x) \tag{3.15}$$

It is easy to check that $\psi \circ \tilde{\phi}$ is Γ equivarient. Thus:

$$\Sigma(A) \leq \Sigma(\psi \circ \tilde{\phi}(A)) \tag{3.16}$$

Note that $|\psi \circ \tilde{\phi}(b)| = R + \epsilon |T(A)|$ and $|\psi \circ \tilde{\phi}(x)| > R$ if and only if $x \in \rho B_\delta(b)$ for some $\rho \in \Sigma(A)$.

Let $\rho \in \Sigma(\psi \circ \tilde{\phi}(A))$. Then there exists $c \in A$ such that $\psi \circ \tilde{\phi}(c) = \rho \psi \circ \tilde{\phi}(b)$. Thus, $|\psi \circ \tilde{\phi}(c)| = R + \epsilon |T(A)| > R$. Thus, there exists $d \in B_\delta(b)$ such that $c = \rho' d$ for some $\rho' \in \Sigma(A)$. Now, if $d \neq b$, we can choose $\delta_1 < \delta$ so that $d \notin B_{\delta_1}(b)$. Then there exists $d_1 \in B_{\delta_1}(b)$ such that $c = {\rho_1}' d$ for some ${\rho_1}' \in \Sigma(A)$. However, because the actions of ${\rho_1}'$ and ρ' are invertible linear transformations, we must have $\rho_1' \neq \rho'$. Consequently, since Γ is finite, we must be able to choose δ small enough that $d = b$. Therefore $\rho\psi \circ \tilde{\phi}(b) = \rho'\psi \circ \tilde{\phi}(b)$. Thus:

$$\rho'^{-1}\rho \in \Sigma(\psi \circ \tilde{\phi}(b)) = T(A) \leq \Sigma(A). \tag{3.17}$$

Consequently, $\rho \in \Sigma(A)$ and $\Sigma(A) = \Sigma(\psi \circ \tilde{\phi}(A)) = \Sigma(\tilde{\phi}(A))$.

By definition of ψ, $\psi(r\hat{s}) = r\psi(\hat{s})$, so

$$\begin{aligned} \tilde{\phi}(x) = & \ \psi^{-1}\left[\left(1 + \epsilon \textstyle\sum_{\gamma\in\Gamma} \eta(\gamma x)\right) \psi \circ \phi(x)\right] \\ = & \ \left(1 + \epsilon \textstyle\sum_{\gamma\in\Gamma} \eta(\gamma x)\right) \phi(x). \end{aligned} \tag{3.18}$$

Thus,

$$\|\tilde{\phi} - \phi\|_{C^k} \leq \epsilon |\Gamma| \, \|\eta\|_{C^k} \|\phi\|_{C^k} \tag{3.19}$$

which can be made as small as possible.

□

Corollary 3.1 *Let Γ act faithfully on $\boldsymbol{M}$ and orthogonally on $\boldsymbol{V}$. Let $\mathcal{A}(\boldsymbol{M})$ be defined as above. Let $A \in \mathcal{A}(\boldsymbol{M})$. If $\boldsymbol{M} \sim \boldsymbol{V}$ then there exists an open dense set $\mathcal{O} \subset C^k_\Gamma(\boldsymbol{M}, \boldsymbol{V})$ such that for all $\phi \in \mathcal{O}$:*

$$\Sigma(A) = \Sigma(\phi(A)) \tag{3.20}$$

Corollary 3.2 *Let Γ act faithfully on $\boldsymbol{M}$ and orthogonally on $\boldsymbol{V}$. Let $\mathcal{A}(\boldsymbol{M})$ be defined as above. Let $A \in \mathcal{A}(\boldsymbol{M})$. If $T(A)$ is an isotropy subgroup for $\boldsymbol{V}$ then there exists an open dense set $\mathcal{O} \subset C^k_\Gamma(\boldsymbol{M}, \boldsymbol{V})$ such that for all $\phi \in \mathcal{O}$:*

$$\Sigma(A) = \Sigma(\phi(A)) \tag{3.21}$$

4 Systems with $\mathbf{D}_n$ and $\mathbf{D}_n \times \mathbf{Z}_2$ symmetry

Two common classes of symmetry groups for differential equations are:

1. $\mathbf{D}_n = \langle \rho, \kappa \mid \rho^n = \kappa^2 = 1, \ \rho\kappa = \kappa\rho^{n-1} \rangle$
2. $\mathbf{D}_n \times \mathbf{Z}_2 \cong \langle \rho, \kappa, \sigma \mid \rho^n = \kappa^2 = \sigma^2 = 1, \ \kappa\sigma = \sigma\kappa, \rho\kappa = \kappa\rho, \rho\kappa = \kappa\rho^{n-1} \rangle$

We define

$$\Gamma_n := \langle \rho, \kappa, \sigma \mid \rho^n = \kappa^2 = \sigma^2 = 1, \ \kappa\sigma = \sigma\kappa, \rho\kappa = \kappa\rho, \rho\kappa = \kappa\rho^{n-1} \rangle \tag{4.1}$$

and consider $\mathbf{D}_n < \Gamma_n$.

It is common for Γ_n to act on $M = \mathbf{R}^n$ as follows:

$$\rho \bullet (x_0, x_1, \ldots, x_{n-1}) = (x_{n-1}, x_0, x_1, \ldots, x_{n-2}) \tag{4.2}$$

$$\kappa \bullet (x_0, x_1, \ldots, x_{n-1}) = (x_0, x_{n-1}, \ldots, x_1) \tag{4.3}$$

$$\sigma \bullet (x_0, x_1, \ldots, x_{n-1}) = (-x_0, -x_1, \ldots, -x_{n-1}) \tag{4.4}$$

Since this action is linear, let ρ_M be the matrix of the action of ρ, κ_M and σ_M similarly. We use the notation $(\rho_M)_{ij}$ for the i, j component of ρ_M where i and j range from 0 to $n-1$.

$$(\rho_M)_{ij} = \begin{cases} 1 \text{ if } i-j \equiv 1 \bmod n \\ 0 \text{ otherwise} \end{cases} \tag{4.5}$$

$$(\kappa_M)_{ij} = \begin{cases} 1 \text{ if } i+j \equiv 0 \mod n \\ 0 \text{ otherwise} \end{cases} \tag{4.6}$$

$$(\sigma_M)_{ij} = -\delta_{ij} \tag{4.7}$$

Where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{4.8}$$

When using matrix notation, we consider elements of M to be column vectors. The matrices operate from the left.

4.1 A Symmetry Preserving Map from M into $\mathbf{C}^n$

Now, consider the linear transformation $\mathcal{Z} : M \rightarrow \mathbf{C}^n$ defined by its matrix:

$$(\mathcal{Z})_{ij} = \frac{1}{\sqrt{n}}\omega^{ij} \tag{4.9}$$

where $\omega = e^{\imath\frac{2\pi}{n}}$.

Note that $(\mathcal{Z}^\dagger)_{ij} = \frac{1}{\sqrt{n}}\omega^{-ij}$, so that:

$$(\mathcal{Z}^\dagger \mathcal{Z})_{ij} = \frac{1}{n}\sum_{k=0}^{n-1}\omega^{-ik}\omega^{kj} = \frac{1}{n}\sum_{k=0}^{n-1}\omega^{k(j-i)} = \delta_{ij} \tag{4.10}$$

I.e., $\mathcal{Z}$ is unitary.

Remark 1 *We have used the fact that if α is an n^{th} root of unity and $\alpha \neq 1$ then*

$$\sum_{i=0}^{n-1}\alpha^i = 0 \tag{4.11}$$

which can easily be verified by noting that:

$$\alpha\sum_{i=0}^{n-1}\alpha^i = \sum_{i=1}^{n}\alpha^i = 1 + \sum_{i=1}^{n-1}\alpha^i = \sum_{i=0}^{n-1}\alpha^i \tag{4.12}$$

So, $\sum_{i=0}^{n-1}\alpha^i = 0$.

Thus $\mathcal{Z}$ is an invertible linear transformation from M into $\mathbf{C}^n$. Let $V = \mathcal{Z}(M)$, *i.e.* the range of $\mathcal{Z}$. There is a natural action of Γ_n induced on V by $\mathcal{Z}$. It is easiest to express this action in terms of the matrices of the action:

$$\rho_V = \mathcal{Z}\rho_M\mathcal{Z}^\dagger \tag{4.13}$$

$$\kappa_V = \mathcal{Z}\kappa_M\mathcal{Z}^\dagger \tag{4.14}$$

$$\sigma_V = \mathcal{Z}\rho_M\mathcal{Z}^\dagger \tag{4.15}$$

We now calculate these matrices:

$$\begin{aligned}(\rho_V)_{ij} &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{h=0}^{n-1}\omega^{ik}(\rho_M)_{kh}\omega^{-hj} = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{h=0}^{n-1}\omega^{ik-hj}(\rho_M)_{kh} \\ &= \frac{1}{n}\sum_{k=0}^{n-1}\omega^{ik-(k-1)j} = \frac{\omega^j}{n}\sum_{k=0}^{n-1}\omega^{k(i-j)} = \omega^j\delta_{ij}\end{aligned} \tag{4.16}$$

$$(\kappa_V)_{ij} = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{h=0}^{n-1}\omega^{ik-hj}(\kappa_M)_{kh} = \begin{cases} 1 & \text{if } i+j \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases} = (\kappa_M)_{ij} \tag{4.17}$$

$$(\sigma_V)_{ij} = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{k=0}^{n-1}\omega^{ik-hj}\delta_{kh} = \frac{1}{n}\sum_{k=0}^{n-1}-\omega^{ik-kj} = -\delta_{ij} = (\sigma_M)_{ij} \tag{4.18}$$

V must have real dimension n, and is thus a proper (real) subspace of $\mathbf{C}^n$. Note that if $z \in V$ then $z = \mathcal{Z}x$ for some $x \in M$ and consequently:

$$z_i = \sum_{j=0}^{n-1}\omega^{ij}x_j = \sum_{j=0}^{n-1}\bar{\omega}^{(n-i)j}\bar{x}_j = \bar{z}_{n-i} \tag{4.19}$$

So V is the subspace of $\mathbf{C}^n$ which is the fixed point subspace of the operator $\mathcal{R}$ defined by:

$$(z_0, z_1, \ldots, z_{n-1}) \overset{\mathcal{R}}{\longmapsto} (z_0, \bar{z}_{n-1}, \ldots, \bar{z}_1) \tag{4.20}$$

$\mathcal{R}$ is not linear with respect to the complex structure of $\mathbf{C}^n$, but it is a *real* linear operator (if $\mathbf{C}^n$ is considered as a real vector space ($\mathbf{R}^{2n}$)).

Note that z_0 is real and that for n even, $z_{n/2}$ is real. This motivates defining the following subspaces of $\mathbf{C}^n$:

$$\begin{aligned} V_0 &= \{z \in \mathbf{C}^n \mid z_0 = \bar{z}_0,\ z_i = 0 \text{ if } i \neq 0\} \\ V_1 &= \{z \in \mathbf{C}^n \mid z_i = 0 \text{ if } i \neq 1\} \\ &\vdots \\ V_{\lfloor\frac{n}{2}\rfloor-1} &= \left\{z \in \mathbf{C}^n \mid z_i = 0 \text{ if } i \neq \left\lfloor\tfrac{n}{2}\right\rfloor - 1\right\} \\ V_{\lfloor\frac{n}{2}\rfloor} &= \begin{cases} \left\{z \in \mathbf{C}^n \mid z_i = 0 \text{ if } i \neq \left\lfloor\tfrac{n}{2}\right\rfloor\right\} & n \text{ odd} \\ \left\{z \in \mathbf{C}^n \mid z_{\frac{n}{2}} = \bar{z}_{\frac{n}{2}},\ z_i = 0 \text{ if } i \neq \tfrac{n}{2}\right\} & n \text{ even} \end{cases}\end{aligned} \tag{4.21}$$

Now consider the action of $\mathcal{R}$ on these subspaces:

$$V = \mathcal{Z}(\mathbf{R}^n) = V_0 \oplus V_1 \oplus \cdots \oplus V_{\lfloor\frac{n}{2}\rfloor} \oplus \begin{cases} \mathcal{R}\left(V_1 \oplus \cdots \oplus V_{\lfloor\frac{n}{2}\rfloor}\right) & n \text{ odd} \\ \mathcal{R}\left(V_1 \oplus \cdots \oplus V_{\frac{n}{2}}\right) & n \text{ even} \end{cases} \tag{4.22}$$

Note that the action of κ on $v \in V$ is to exchange v_i with v_{n-i}. However $v_i = \bar{v}_{n-i}$, so $\kappa \bullet v = \bar{v}$. The actions of ρ and σ on V are diagonal. If we take

$$V' = V_0 \oplus V_1 \oplus \cdots \oplus V_{\lfloor \frac{n}{2} \rfloor} \tag{4.23}$$

then the action of Γ_n on this space is well defined. Note that the map $\mathcal{Z}' : M \to V'$ defined by $\mathcal{Z}' = \mathcal{P} \circ \mathcal{Z}$, where $\mathcal{P} : V \to V'$ projects V onto V' by suppressing V_i for $\lfloor \frac{n}{2} \rfloor < i \leq n-1$, is invertible *so it preserves symmetries.* For theoretical purposes it is easier to work with $\mathcal{Z}$, but for practical applications $\mathcal{Z}'$ provides all of the necessary information.

Note that $\mathcal{Z}'$ can be represented by a complex $\lfloor \frac{n}{2} \rfloor \times n$ matrix:

$$\mathcal{Z} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \omega & \cdots & \cdots & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \cdots & \cdots & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & & & & \vdots \\ 1 & \omega^{\lfloor \frac{n}{2} \rfloor} & \cdots & \cdots & \cdots & \omega^{\lfloor \frac{n}{2} \rfloor} \end{pmatrix} \tag{4.24}$$

The action of Γ_n on V' is given by:

$$\kappa \bullet v = \bar{v} \tag{4.25}$$

$$\rho \bullet v = (v_0, \omega v_1, \omega^2 v_2, \ldots, \omega^{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor}) \tag{4.26}$$

$$\sigma \bullet v = -v \tag{4.27}$$

which is linear with respect to the real structure of V' but not linear with respect to the complex structure.

4.2 Isotropy Equivalent Representations of $\mathbf{D}_n$ and Γ_n

Notice that because the action of Γ_n on V' is diagonal, there is an action induced on the spaces V_p where $0 \leq p \leq \lfloor \frac{n}{2} \rfloor$. Each of these V_p is a representation of Γ_n. Given V_p the action of Γ_n on $v \in V_p$ is:

$$\rho \bullet v_p = \omega^p v_p \tag{4.28}$$

$$\kappa \bullet v_p = \bar{v}_p \tag{4.29}$$

$$\sigma \bullet v_p = -v_p \tag{4.30}$$

Theorem 4.1 *V_p and V_q are isotropy equivalent representations of Γ_n if and only if $\gcd(p,n) = \gcd(q,n)$. The same result holds if Γ_n is replaced by $\mathbf{D}_n$.*

We present a proof of this theorem for V_p and V_q representations of $\mathbf{D}_n$. The proof is very similar for Γ_n but requires a few more cases. We will use the following facts in the proof of the theorem:

1. Let $d = \gcd(p,n)$. The only elements of $\mathbf{D}_n$ which act trivially on V_p are powers of $\rho^{\frac{n}{d}}$.

2. $\mathrm{Fix}_{V_p}(\langle \rho^a \kappa \rangle) = \left\{ re^{i\frac{pa\pi}{n}} \mid r \in \mathbf{R} \right\}$

3. Let $G \leq \mathbf{D}_n$. Then one of the following three items must be true:

   (a) $\mathrm{Fix}_{V_p}(G) = 0$.

   (b) $\mathrm{Fix}_{V_p}(G) = \left\{ re^{i\frac{pa\pi}{n}} \mid r \in \mathbf{R} \right\}$ for some $a \in \mathbf{Z}$, $0 \leq a < n$.

   (c) $\mathrm{Fix}_{V_p}(G) = V_p$.

Proof:(of Theorem 4.1) Let $H \leq G \leq \mathbf{D}_n$. Using Lemma 2.4, it suffices to show that

$$\mathrm{Fix}_{V_p}(G) = \mathrm{Fix}_{V_p}(H) \Rightarrow \mathrm{Fix}_{V_q}(G) = \mathrm{Fix}_{V_q}(H). \tag{4.31}$$

We consider three cases:

1. $\mathrm{Fix}_{V_p}(G) = \mathrm{Fix}_{V_p}(H) = V_p$. Then both G and H consist of elements that act trivially on V_p. Note that elements of $\mathbf{D}_n$ act trivially on V_p if and only if they act trivially on V_q. Thus $\mathrm{Fix}_{V_q}(G) = \mathrm{Fix}_{V_q}(H)$.

2. $\mathrm{Fix}_{V_p}(G) = \mathrm{Fix}_{V_p}(H) = \left\{ re^{i\frac{pa\pi}{n}} \mid r \in \mathbf{R} \right\}$ for some $0 \leq a < n$. Then G and H consist only of reflections and elements that act trivially. Suppose that one of the reflections is $\rho^b \kappa$ for some $0 \leq b < n$. Then $\mathrm{Fix}_{V_q}(G) \subseteq \mathrm{Fix}_{V_q}(H) \subseteq \left\{ re^{i\frac{qb}{n}} \right\}$. Now, $\mathrm{Fix}_{V_q}(G) \neq 0$, since G contains only reflections and elements that act trivially on V_q. Thus $\mathrm{Fix}_{V_q}(G) = \mathrm{Fix}_{V_q}(H)$.

3. $\mathrm{Fix}_{V_p}(G) = \mathrm{Fix}_{V_p}(H) = \{0\}$. Then there is some rotation in H which does not act trivially. This rotation does not act trivially in V_q. Thus, $\mathrm{Fix}_{V_q}(G) = \mathrm{Fix}_{V_q}(H) = \{0\}$

□

Corollary 4.1 *Let $\mathcal{P}_n = \{p \in \mathbf{Z}^+ \mid p \mid n\}$. Then:*

1. *Considering the action of Γ_n:*

$$\mathbf{R}^n \sim V_0 \oplus \bigoplus_{p \in \mathcal{P}_n} V_p \tag{4.32}$$

2. *Considering the action of $\mathbf{D}_n$:*

$$\mathbf{R}^n \sim \bigoplus_{p \in \mathcal{P}_n} V_p \tag{4.33}$$

Proof: Use the map $\mathcal{Z}'$ defined in the previous section and apply Lemma 2.5.
□

4.3 Examples

We provide two examples of systems which could be analyzed by the means outlined herein. We have not completed an analysis to date, but examples will appear in a forthcoming paper.

1. The following system is introduced in [2]. It is a differential equation on $\mathbf{R}^6$ which is considered to be $\mathbf{R}^3 \oplus \mathbf{R}^3$. $\mathbf{D}_3$ acts on each copy of $\mathbf{R}^3$ as shown above. The equations are:

$$\begin{cases} \dot{x}_i = & y_i + \delta x_i^2 y_i \\ \dot{y}_i = & x_i - (x_i^2 - \lambda) y_i + \alpha(y_{i-1} - 2y_i + y_{i+1}) + \beta x_i y_i \end{cases} \tag{4.34}$$

   δ, λ, and β are parameters. The equations have $\mathbf{D}_3$ symmetry. Thus, by projecting an attractor into the spaces V_1 and we should be able to (in most cases) analyze the symmetry.

2. The following system is analyzed partially in [7]. It is a differential equation in $\mathbf{R}^{12} = \mathbf{R}^6 \oplus \mathbf{R}^6$. The symmetries of the equations are Γ_6. Thus, by projecting the attractors into $V_0 \oplus V_1 \oplus V_2$, we should be able to (in most cases) analyze the symmetry.

$$\begin{cases} \dot{x}_i = & -4x_i + y_i \\ & +(x_i^2 + y_i^2)(Px_i - Qy_i) - \lambda(4(x_{i-1} + x_{i+1}) - 2(y_{i-1} + y_{i+1})) \\ \dot{y}_i = & -x_i - 4y_i \\ & +(x_i^2 + y_i^2)(Px_i + Qy_i) - \lambda(2(x_{i-1} + x_{i+1}) - 4(y_{i-1} + y_{i+1})) \end{cases} \tag{4.35}$$

   P, Q, and λ are paramaters.

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