# On the Edge-Transitivity of Circulant Graphs

by

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## Abstract

We define and study circulant graphs, and prove Adam's isomorphism theorem. We then define edge-transitivity and enumerate several families of edge-transitive circulants, including s-partite graphs, wreath graphs and cardinal products. We conclude with speculation on the edge-transitivity of circulants with a prime number of vertices.

**Section 1. Introduction**

This work will focus on a class of simple, undirected graphs known as circulants. A *circulant graph*, written $C_p(a_1,a_2,\ldots,a_k)$ or simply $C_p(a_i)$, has vertex set $0,\ldots,p-1$ and edges $(u,v)$ iff $\pm(u-v) \equiv a_i \pmod{p}$ for some $0 \le i \le k$ **[Bo]** . The vertices are usually arranged in order around a circle (note: the vertex 0 can be replaced with p). The $a_i$ are called the *jumps,* and together $a_1,a_2,\ldots,a_k$ form the *jump sequence.* If a jump $a_i$ is greater than p/2, then it is equivalent to the jump $p-a_i$ and is usually reduced to the latter form; so $C_7(6,5) = C_7(1,2)$. An edge which is created due to a jump of $a_i$ is said to be a $a_i$-edge, or to belong to the $a_i$-class. An example of a circulant is $C_7(1,2)$ which is depicted below:

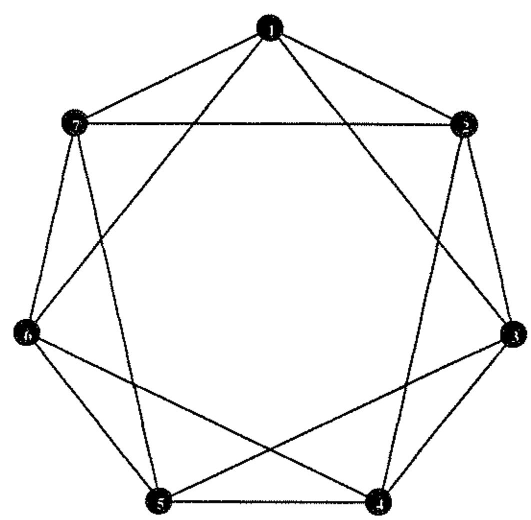


Figure 1: $C_7(1,2)$

If p is even and $a_k = p/2$, then $a_k$ is called a *diagonal jump* and each vertex is incident to exactly one $a_k$-edge. If $a_i < p/2$, then each vertex is incident to exactly two $a_i$-edges. The *degree* of a vertex v is the number of edges incident on v. A circulant without a diagonal jump has vertices of degree 2k, while a circulant with a diagonal jump has vertices of degree 2k-1. The set $A_p$ is defined as the canonical set of possible jumps for a circulant with p vertices; thus $A_p = \{1,2,\ldots,\lfloor p/2 \rfloor\}$. For example, $A_4 = A_5 = \{1,2\}$.

A *symmetry* (automorphism) of a graph G is an isomorphism from G onto itself, and can be thought of as a permutation on the vertices which preserves adjacency. These symmetries form a group under composition. Two vertices (or edges) are *similar* if there exists a symmetry which takes one to the other. A graph is *vertex-transitive* if all pairs of vertices are similar, and is *edge-transitive* if all pairs of edges are similar. A *dart* is a vertex and one of its incident edges and is represented by the notation (u,v) where the vertex u and the edge (u,v) comprise the dart. Thus the dart (u,v) is different from the dart (v,u) and can therefore be thought of as a directed edge. However it is important to note that this does not imply that the graph is directed. A graph is *dart-transitive* if all pairs of darts are similar.

**Lemma 1.1**: All circulants have as symmetries the rigid motions of rotation and reflection.

**Proof:**

(1) Define r: $\{0,1,\ldots,p-1\} \rightarrow \{0,1,\ldots,p-1\}$ by $r(x) = x+c \pmod p$ for some integer c. Then r is a rotation on $C_p(a_i)$.

The vertex u is adjacent to the vertex v in $C_p(a_i)$

iff $\pm(u-v) \equiv a_i \pmod p$ for some $0 \le i \le k$.

iff $\pm(r(u)-r(v)) = \pm(u+c-v-c) = \pm(u-v) \equiv a_i \pmod p$ for some $0 \le i \le k$.

Thus r(u) is adjacent to r(v). Since r is a permutation of the vertices which preserves adjacency, rotation is a symmetry on $C_p(a_i)$.

(2) Define s: $\{0,1,\ldots,p-1\} \rightarrow \{0,1,\ldots,p-1\}$ by $s(x) = c-x \pmod p$ for some integer c. Note that s is a reflection on $C_p(a_i)$.

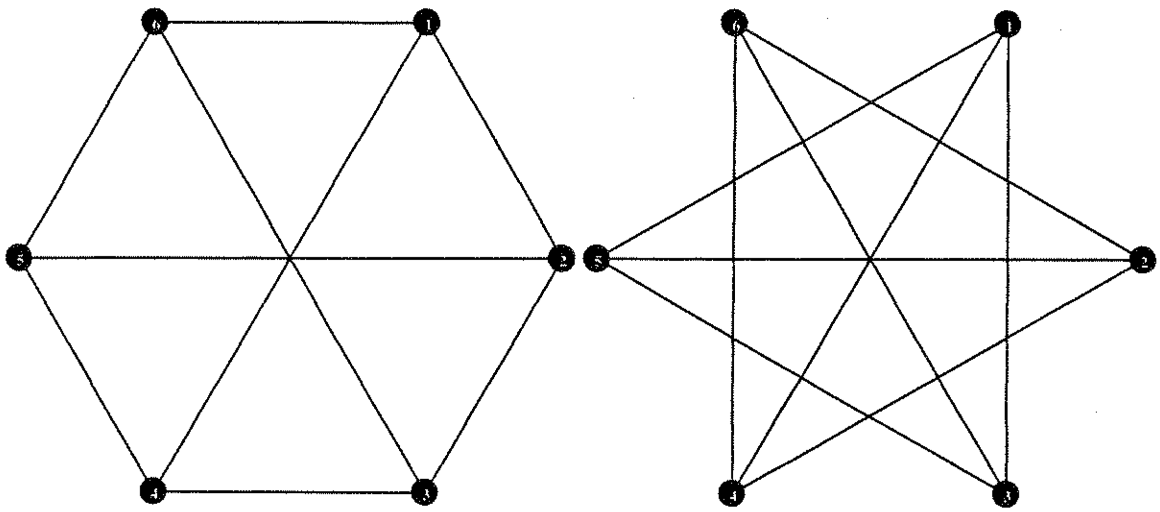
The vertex u is adjacent to the vertex v in $C_p(a_i)$

iff $\pm(u-v) \equiv a_i \pmod p$ for some $0 \le i \le k$.

iff $\pm(s(u)-s(v)) = \pm(c-u-c+v) = \pm(v-u) = \pm(u-v) \equiv a_i \pmod p$ for some $0 \le i \le k$.

Thus s(u) is adjacent to s(v). Since s is a permutation of the vertices which preserves adjacency, reflection is a symmetry on $C_p(a_i)$.

Any two vertices of a circulant are similar through rotation, thus all circulants are vertex-transitive. However, circulants may or may not be edge-transitive. For example in Figure 2, $C_6(1,3)$ is edge-transitive while $C_6(2,3)$ is not.



$C_6(1,3)$
Edge-transitive

$C_6(2,3)$
Not Edge-transitive

Figure 2

It is easy to see that $C_6(2,3)$ fails to be edge-transitive because symmetries preserve cycles. Since the edge (6,2) is in a 3-cycle but (6,3) is not, these edges are not similar, proving that the graph is not edge-transitive. To show that $C_6(1,3)$ is edge-transitive, the following lemma is useful.

**Lemma 1.2**: The graph $C_p(a_1,a_2,\ldots,a_k)$ is edge-transitive iff for each pair of jumps $a_i$ and $a_{i+1}$ there exists a symmetry which takes an $a_i$-class edge to an $a_{i+1}$-class edge.

**Proof**: Any two edges in the same jump class are similar by rotation. Let the symmetry $\sigma$ guaranteed by the hypothesis take $(j,j+a_i)$ from the $a_i$-class to $(k,k+a_{i+1})$ of the $a_{i+1}$-class. Since the composition of multiple symmetries is again a symmetry , similarity between edges (and vertices and darts) is transitive. Thus any edge in the $a_i$-class is similar to $(j,j+a_i)$ which is similar by $\sigma$ to $(k,k+a_{i+1})$ which in turn is similar to all edges in the $a_{i+1}$-class. Thus any two edges of adjacent jump classes are similar, however by transitivity this implies that any two edges of any two jump classes are similar. Therefore since any pair of edges is similar, the circulant is edge-transitive.

Thus, in order to show that $C_6(1,3)$ is edge-transitive, it is sufficient to find a symmetry which takes a 1-class edge to a 3-class edge. The symmetry f = (1 3)(2 4) takes the edge (0,1) to the edge (0,3). So, by Lemma 1.1, $C_6(1,3)$ is edge-transitive.

**Lemma 1.3**: If a circulant is edge-transitive, then it is dart-transitive.

**Proof**: All circulants have as a symmetry the reflection s which will take the dart (0,1) to the dart (1,0). Consider the dart (u,v). Since the circulant is edge-transitive, the edge (u,v) is similar to the edge (0,1). Thus the dart (u,v) is similar to either the dart (0,1) or (1,0). Since all darts are similar to either (0,1) or (1,0) which are similar to each other, any pair of darts are similar. Therefore the graph is dart-transitive.

The preceding lemma is a useful property of circulants which will be used in a later proof. Also useful is the fact that the complement of a circulant is again a circulant.

The *complement* of a graph G is the graph $\overline{G}$ with the same vertices as G. An edge exists in $\overline{G}$ iff it does not exist in G. Thus $\overline{C_p(a_1,a_2,\ldots,a_k)} = C_p(\{A_p \setminus \{a_1,a_2,\ldots,a_k\}\})$.

## Section 2. Isomorphism

One of the natural questions raised when examining graphs is of isomorphism. Is it possible to determine just by exploring the notation when two graphs are isomorphic? Obviously, to be isomorphic two circulants must have the same number of vertices and edges (which correspond to jumps for circulants). Also, a circulant with a diagonal jump cannot be isomorphic to a circulant which does not have a diagonal jump. The following theorem provides for a case when two circulants must be isomorphic.

**Theorem 2.1**: Let gcd(p,n) = 1 (p and n are relatively prime). Then $C_p(a_1,a_2,\ldots,a_k) \cong C_p(na_1,na_2,\ldots,na_k)$, where multiplication is mod p.

**Example**: Note that gcd(8,3) = 1 and that $C_8(1,2) \cong C_8(2,3)$ since 3*1 = 3 and 3*2 = 6 which is reduced to 8-6 which equals 2.

**Proof**: Let $G = C_p(a_1,a_2,\ldots,a_k)$ and $G' = C_p(na_1,na_2,\ldots,na_k)$. Define $\phi : \{0,1,\ldots,p-1\} \rightarrow \{0,1,\ldots,p-1\}$ by $\phi(x) = nx \pmod{p}$. Since gcd(p,n) = 1, $\phi$ is a bijection. Two vertices j and k are adjacent in G iff $\pm (j-k) \equiv a_i \pmod{p}$. Again, since n is relatively prime to p, this is equivalent to $\pm (nj-nk) \equiv a_i n \pmod{p}$. Thus $\phi(j)$ is adjacent to $\phi(k)$ in G′. Therefore $\phi$ is a 1-1 function on the set of vertices which preserves adjacency, and is thus an isomorphism.

This theorem was originally proved by Adam in 1967, although we are not aware of his method of proof. Thus, two circulants which are isomorphic due to this theorem are called Adam isomorphic. We, like Adam, originally speculated that the converse of this theorem was true: that any two isomorphic circulants must be Adam isomorphic. We were both wrong. Elspas and Turner showed in 1970 that $C_{16}(1,2,7) \cong C_{16}(2,3,5)$, which are not Adam isomorphic as there is no integer n which when multiplied by the jump sequence (1,2,7) results in (2,3,5) [El] . Thus, this theorem allows us to show that two circulants are isomorphic, but just because an n cannot be found which fulfills the hypothesis does not mean that the graphs are not isomorphic.

## Section 3. Basic Families of Edge-Transitive Circulants

Thus, we are brought to the main body of this work: finding families of edge-transitive circulants. The section for each family concludes with an associated formula, labelled (1) through (11), which will produce an edge-transitive circulant. These families are not exhaustive, so it is not necessary for an edge-transitive circulant to belong to one of the families presented in this paper. It is doubtful that an exhaustive categorization could be found without a deeper understanding of the symmetries which guarantee each graph's edge-transitivity. With this said, we will start with a few basic cases.

**Theorem 3.1:** All cycle graphs are edge-transitive circulants of the form $C_p(n)$ where gcd(p,n) = 1.

**Example:** The pentagon, $C_5(1)$, which is isomorphic to the five-pointed star, $C_5(2)$, is edge-transitive.

**Proof:** It is obvious that all cycle graphs are circulants of the form $C_p(1)$, which is Adam isomorphic to all $C_p(n)$ where n is relatively prime to p. Since all of the edges of a cycle graph are in the same jump class, all pairs of edges are similar by rotation. Therefore all cycle graphs are edge-transitive.

Theorem 3.1 gives us our first edge-transitive family of circulants:

$$C_p(n) \text{ where } \gcd(p,n) = 1 \tag{1}$$

Cycle graphs on p vertices are abbreviated simply as $C_p$.

**Theorem 3.2:** All complete graphs are edge-transitive circulants.

**Proof:** In the complete graph, each vertex is adjacent to every other vertex. It is written $C_p(A_p)$, or abbreviated $K_p$. Since each vertex is adjacent to every other vertex, any permutation of the vertices will preserve adjacency and thus will be a symmetry. Thus any edge (j,k) is similar to any other edge (u,v) since any permutation f such that f(j) = u and f(k) = v is a symmetry which takes one edge to the other. Since all pairs of edges are similar, $K_p$ is edge-transitive.

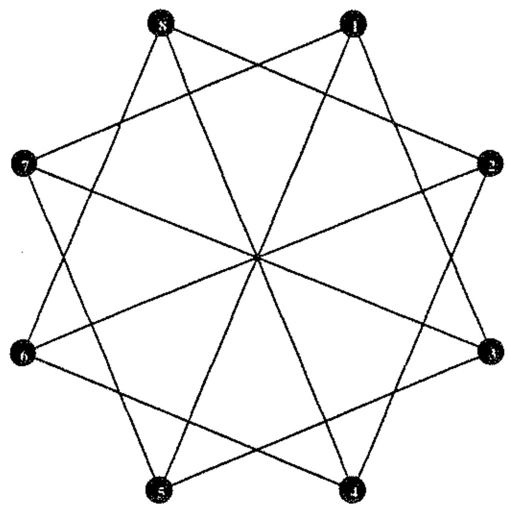
Thus our second family of edge-transitive circulants is:

$$K_p = C_p(A_p) \tag{2}$$

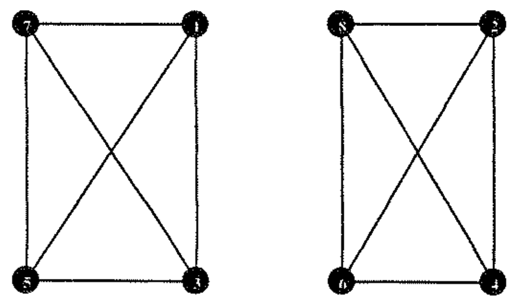
If $\gcd(p, a_1, a_2, \ldots, a_k) = d \neq 1$, then the circulant $C_p(a_1, a_2, \ldots, a_k)$ is called a *multiple graph*. The following theorem shows that this is a disconnected graph.

**Theorem 3.3**: Let $\gcd(p, a_1, a_2, \ldots, a_k) = d \neq 1$. Then $C_p(a_1, a_2, \ldots, a_k) \cong d\,C_{p/d}(a_1/d, a_2/d, \ldots, a_k/d)$ which is notation for d separate copies of $C_{p/d}(a_1/d, a_2/d, \ldots, a_k/d)$.

**Example**: $C_8(2,4) \cong 2\,C_4(1,2)$:



$C_8(2,4)$



$2C_4(1,2)$

Figure 3

**Proof:**

(1) Examine two adjacent vertices u and v in $C_p(a_1,a_2,\ldots,a_k)$. We know that $\pm(u-v) \equiv a_i \pmod p$ for some $1 \le i \le k$. Since $d|p$ and $d|a_i$ then $u-v \equiv 0 \pmod d$. This is an equivalence relation which partitions the set of vertices into d classes:

$\{0,d,\ldots,p-d\}$, $\{1,d+1,\ldots,p-d+1\}$, . . . , $\{d-1,2d-1,\ldots,p-1\}$.

No edge exists between two vertices not in the same class, thus we have d separate graphs.

2) Examine the graph on the vertices $\{j,d+j,\ldots,p-d+j\}$. Two vertices b and c are adjacent if $\pm(b-c) \equiv a_i \pmod p$. Since d divides all the vertices - j,

$$\pm((b-j)/d - (c-j)/d) = \pm(b/d - c/d) \equiv a_i/d \pmod{p/d}.$$

So if we renumber each vertex u by $(u-j)/d$, we have the graph on $p/d$ vertices labelled $\{0,1,\ldots,p/d-1\}$ which are adjacent only if their difference is congruent to some $a_i/d \pmod{p/d}$. Thus it is the graph $C_{p/d}(a_1/d,a_2/d,\ldots,a_k/d)$, which is called the *base graph*. Therefore $C_p(a_1,a_2,\ldots,a_k)$ is made up of d separate copies of the base graph which $= d\,C_{p/d}(a_1/d,a_2/d,\ldots,a_k/d)$.

**Theorem 3.4**: If $\gcd(p, a_1, a_2, ..., a_k) = d \neq 1$ and $C_{p/d}(a_1/d, a_2/d, ..., a_k/d)$ is edge-transitive then $C_p(a_1, a_2, ..., a_k)$ is edge-transitive.

**Proof**: Since the base graph $C_{p/d}(a_1/d, a_2/d, ..., a_k/d)$ is edge-transitive, all pairs of edges within a base graph are similar. Rotation will take one base graph to all of the others in the multiple graph, thus the base graphs are similar. Thus all pairs of edges within the multiple graph are similar making the multiple graph $C_p(a_1, a_2, ..., a_k)$ edge-transitive.

This gives us our third family:

if $\gcd(p, a_1, a_2, ..., a_k) = d \neq 1$ and $C_{p/d}(a_1/d, a_2/d, ..., a_k/d)$ is edge-transitive then $C_p(a_1, a_2, ..., a_k)$ is edge-transitive. (3)

**Theorem 3.5**: For $C_p(a_1, a_2, ..., a_k)$, if $\gcd(n,p) = 1$ and n cyclically permutes the jumps (meaning each $a_i = n^j a_1$ for some $1 \leq j \leq k$), then the circulant is edge-transitive. Also $n^k \equiv \pm 1 \pmod{p}$.

**Example**: Consider $C_{15}(1,2,4,7)$ with $n = 2$. Note that $\gcd(15,2) = 1$ and $1*2 = 2$, $2*2 = 4$, $4*2 = 8$ which reduces to a jump of 7, and $2*7 = 14$ which reduces to a jump of 1. Also, $2^4 = 16 \equiv 1 \pmod{15}$.

**Proof**:

(1) From Lemma 1.2, to prove that $C_p(a_1, a_2, ..., a_k)$ is edge-transitive, it is sufficient to show that for any two jumps $a_i$ and $a_{i+1}$ there exists a symmetry which takes an $a_i$-class edge to an $a_{i+1}$-class edge. Since n cyclically permutes the jumps, $a_{i+1} = n^j * a_i$ for some $j<k$. Two vertices u and v are adjacent iff $\pm(u-v) \equiv a_i \pmod{p}$. Since n is relatively prime to p, multiplying each side of the equation by $n^j$ will result in $\pm(n^j u - n^j v) \equiv a_i n^j \pmod{p} = a_{i+1} \pmod{p}$. If $V = \{\text{vertices of } C_p(a_1, a_2, ..., a_k)\}$, we can define $\phi: V \to V$ to be $\phi(x) = n^j x \pmod{p}$. Since $\pm(\phi(a_i) - \phi(a_{i+1})) \equiv a_{i+1} \pmod{p}$, $\phi(a_i)$ and $\phi(a_{i+1})$ are adjacent. Therefore $\phi$ is a symmetry which takes an $a_i$-class edge to an $a_{i+1}$-class edge

and $C_p(a_1, a_2, \ldots, a_k)$ is edge-transitive.

(2) Since n cyclically permutes the jump sequence, $a_i * n^k \equiv a_i \pmod{p}$. Therefore $n^k \equiv 1 \pmod{p}$.

Thus we are presented with the following broad family of edge-transitive circulants:

$$C_p(a_1, a_2, \ldots, a_k) \text{ where } \gcd(n,p) = 1 \text{ and } n \text{ cyclically permutes the jumps} \quad (4)$$

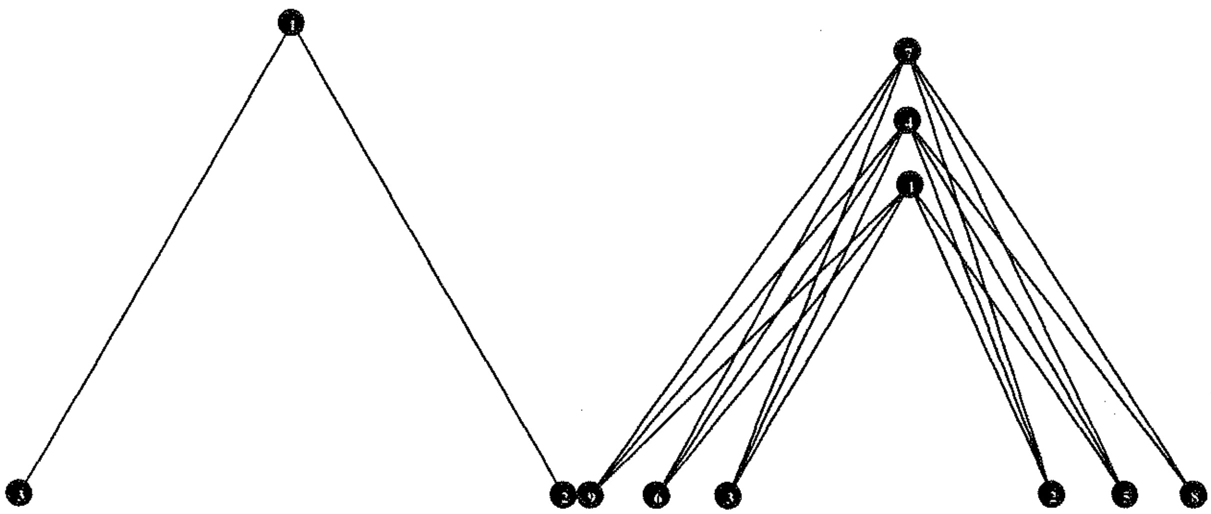
### Section 4. Set Product

**Definition 4.1**: The *set product* of a graph $G_1$ and an integer n, written as $G_1 \sim n$, is defined as follows:

The vertices of $G \sim n = V(G) \times \{0,\ldots,n-1\}$

The edges of $G \sim n = \{((a,b),(c,d)) \mid (a,c) \text{ is an edge in } G_1\}$

**Example**:



$G_1$ $\qquad$ $G_1 \sim 3$

Figure 4

**Theorem 4.1**: If a graph G is edge-transitive, then G ~ n is edge-transitive.

**Proof**: Each edge in G is replaced by a "clump" of edges in G ~ n. We will first show that all clumps of edges in G ~ n are similar. Also, all edges within one clump are similar. Thus by transitivity, any two edges in G ~ n are similar. More specifically, let [(a,i),(b,j)] and [(c,k),(d,l)] be two edges in G ~ n. We must show that there exists a symmetry which takes one to the other.

First, we will show that there exists a symmetry which takes [(a,i),(b,j)] to [(c,i),(d,j)]. By definition of G ~ n, we know that (a,b) and (c,d) are edges in G. Since G is edge-transitive, there exists a symmetry $\phi$:V(G) $\longrightarrow$ V(G) which takes (a,b) to (c,d). Without loss of generality, we will assume $\phi$(a) = c and $\phi$(b) = d. Define $\phi'$: V(G ~ n) $\longrightarrow$ V(G ~ n) by $\phi'$((a,i)) = ($\phi$(a),i). Must show that $\phi'$ is a symmetry.

The vertices (x,m) , (y,n) are adjacent in G ~ n
iff x and y are adjacent in G
iff $\phi$(x) and $\phi$(y) are adjacent in G
iff ($\phi$(x),m) and ($\phi$(y),n) are adjacent in G ~ n
iff $\phi'$(x,m) and $\phi'$(y,n) are adjacent in G ~ n.

Since $\phi'$ is a permutation of the vertices which preserves adjacency, $\phi'$ is a symmetry. Since $\phi'$((a,i)) = (c,i) and $\phi'$((b,j)) = (d,j), the edge [(a,i),(b,j)] is similar to [(c,i),(d,j)].

Second, we will show that there exists a symmetry which takes [(c,i),(d,j)] to [(c,k),(d,l)] which are in the same clump of edges in G ~ n. Note that the vertices (c,i) and (c,k) are both adjacent to the set of vertices {u in G | u is adjacent to c} x {0,1,...,n-1}. Thus the permutation which interchanges (c,i) and (c,k) and fixes all other vertices is a symmetry. For the same reason, there is a symmetry which interchanges (d,j) and (d,l). The composition of these two symmetries takes the edge [(c,i),(d,j)] to [(c,k),(d,l)], thus the two edges are similar.

By transitivity, [(a,i),(b,j)] is similar to [(c,k),(d,l)]. Thus any pair of edges in G ~ n is similar. Therefore G ~ n is edge-transitive.

A *complete s-partite graph* is written $K_{n,n,\ldots,n}$ where s is the number of n's. There are s sets of n vertices, and a vertex of one set is adjacent to every vertex in every other set. For example, figure 5 shows $K_{2,2,2,2}$.

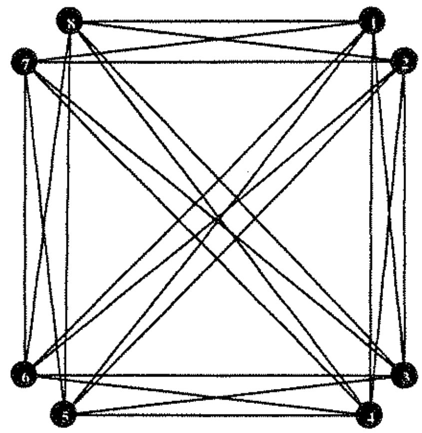


Figure 5: $K_{2,2,2,2}$

**Theorem 4.2**: Complete s-partite graphs are edge-transitive circulants, with $K_{n,n,\ldots,n} = C_{sn}(\{a \in A_{sn} \mid s \text{ does not divide } a\})$

**Example:** $K_{5,5,5} = C_{15}(1,2,4,5,7)$

**Proof**: Note that $K_{n,n,\ldots,n} = K_s \sim n$. $K_s$ is edge-transitive from Theorem 3.2, so from Theorem 4.1, $K_s \sim n = K_{n,n,\ldots,n}$ is edge-transitive. It remains to be shown that complete s-partite graphs are circulant. Notice that in the complement of $K_{n,n,\ldots,n}$, each vertex is adjacent only to all of the vertices in its own set, forming s separate copies of the complete graph on n vertices.

$$\begin{aligned} \overline{K_{n,n,\ldots,n}} &= sK_n \\ &= sC_n(A_n) \\ &= C_{sn}(s,2s,\ldots,\lfloor n/2 \rfloor s) \text{ by Theorem 3.3.} \\ K_{n,n,\ldots,n} &= C_{sn}(\{a \in A_{sn} \mid s \text{ does not divide } a\}). \end{aligned}$$

Thus we have another family of edge-transitive circulants:

$$C_{sn}(\{a \in A_{sn} \mid s \text{ does not divide } a\}) \qquad (5)$$

*Wreath graphs*, written W(s,n), also have s sets of n vertices. The sets of vertices are arranged around a circle, and a vertex in one set is adjacent to the vertices of the sets immediately to the left and right of its own set. For example, W(4,2) is displayed in figure 6.

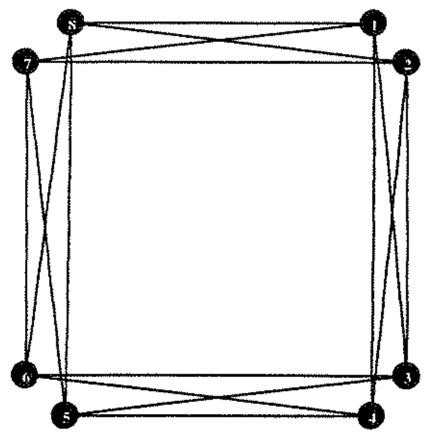


Figure 6: W(4,2)

**Theorem 4.3**: All wreath graphs are edge-transitive circulants, with $W(s,n) \cong C_{sn}(\{a \in A_{sn} \mid a \equiv \pm 1 \pmod{s}\})$

**Example**: $W(5,3) \cong C_{15}(1,4,6)$

**Proof**: Note that $W(s,n) \cong C_s \sim n$. We know that $C_s$ is edge-transitive from Theorem 3.1, so from Theorem 4.1, $C_s \sim n \cong W(s,n)$ is edge-transitive. It remains to be shown that wreath graphs are circulant.

Let W(s,n) consist of vertices $V_{ij}$ where $0 \leq i \leq s-1$ and $0 \leq j \leq n-1$ and edges $(V_{ij}, V_{kl})$ when $k \equiv i \pm 1 \pmod{s}$. Note that i determines which set the vertex is in while j determines the element within the set. Define $\phi$: vertices(W(s,n)) $\rightarrow \{0,1,\ldots,ns-1\}$ to be $\phi(V_{ij}) = i + js$. Let $(V_{ab}, V_{cd})$ be an edge in W(s,n). this means $c \equiv a \pm 1 \pmod{s}$. This is true iff $c - a \equiv \pm 1 \pmod{s}$

iff $c - a + s(d - b) \equiv \pm 1 \pmod{s}$

iff $(c + ds) - (a + bs) \equiv \pm 1 \pmod{s}$

iff $\phi(V_{ab}) - \phi(V_{cd}) \equiv \pm 1 \pmod{s}$

iff $(\phi(V_{ab}),\phi(V_{cd}))$ is an edge in $C_{sn}(\{a \in A_{sn} \mid a \equiv \pm 1 \pmod{s}\})$.
Since $\phi$ is a one-to-one onto function which preserves adjacency, it is an isomorphism.

This proof gives us the following family of edge-transitive circulants:

$$C_{sn}(\{a \in A_{sn} \mid a \equiv \pm 1 \pmod{s}\}) \tag{6}$$

Thus we have shown, using complete s-partite graphs and wreath graphs that the set product of complete graphs and cycle graphs are edge-transitive circulants. This raises the question of the set products of other edge-transitive graphs. It turns out that the set product of any edge-transitive circulant is again an edge-transitive circulant.

**Theorem 4.4**: If $C_p(a_1,a_2,\ldots,a_k)$ is edge-transitive, then $C_p(a_1,a_2,\ldots,a_k) \sim n$ is edge-transitive and is isomorphic to $C_{np}(\{a \in A_{np} \mid a \equiv \pm a_i \pmod{p}\})$.

**Example:** $C_{15}(2,7) \sim 2 \cong C_{30}(2,7,8,13)$ is edge-transitive.

**Proof:** It was shown in Theorem 4.1 that $C_p(a_1,a_2,\ldots,a_k) \sim n$ is edge-transitive when $C_p(a_1,a_2,\ldots,a_k)$ is, so it remains to be shown that $C_p(a_1,a_2,\ldots,a_k) \sim n \cong C_{np}(\{a \in A_{np} \mid a \equiv \pm a_i \pmod{p}\})$

Let $C_p(a_i) \sim n$ consist of vertices (i,j) where $0 \leq i \leq p-1$ and $0 \leq j \leq n-1$ and edges $[(i,j),(k,l)]$ when $k \equiv i \pm a_i \pmod{p}$. Define $\phi$: vertices$(C_p(a_i) \sim n) \to \{0,1,\ldots,np-1\}$ to be $\phi(i,j) = i + jp$. Let $[(a,b),(c,d)]$ be an edge in $C_p(a_i) \sim n$. This means $c \equiv a \pm a_i \pmod{p}$. This is true iff $c - a \equiv \pm a_i \pmod{p}$

iff $c - a + p(d - b) \equiv \pm a_i \pmod{p}$

iff $(c + dp) - (a + bp) \equiv \pm a_i \pmod{p}$

iff $\phi(a,b) - \phi(c,d) \equiv \pm a_i \pmod{p}$

iff $(\phi(a,b),\phi(c,d))$ is an edge in $C_{np}(\{a \in A_{np} \mid a \equiv \pm a_i \pmod{p}\})$.

Since $\phi$ is a one-to-one onto function which preserves adjacency, it is an isomorphism.

Theorem 4.4 gives the following broad family:

$$\text{If } C_p(a_1, a_2, \ldots, a_k) \text{ is edge-transitive, then so is } C_{np}(\{a \in A_{np} \mid a \equiv \pm a_i \pmod p\}) \qquad (7)$$

Note that family 7 incudes families 5 and 6, however families 5 and 6 are useful since they are easier to recognize.

## Section 5. Cardinal Product

The next three families all involve the cardinal (Kronecker) product which is introduced here.

**Definition 5.1**: The *cardinal product* of two graphs $G_1$ and $G_2$, written as $G_1 \wedge G_2$, is defined as follows:

The vertices of $G_1 \wedge G_2 = V(G_1) \times V(G_2)$

The edges of $G_1 \wedge G_2 = \{((a,b),(c,d)) \mid (a,c)$ is an edge in $G_1$ and $(b,d)$ is an edge in $G_2\}$

**Lemma 5.2**: The cardinal product of two edge-transitive circulants is edge-transitive.

**Proof**: Gary Amende showed in his 1994 REU paper that the cardinal product of a dart-transitive graph and an edge-transitive graph is edge-transitive **[Am]**. It was shown in Lemma 1.3 that an edge-transitive circulant must be dart-transitive. Therefore the cardinal product of two edge-transitive circulants is edge-transitive.

For convenience, we will display the vertex set of the cardinal product of two circulants $C_m(a_i)$ and $C_n(b_j)$ in the following chart, where each ordered pair is a vertex of the product graph:

| | 0 | 1 | 2 | ... | j | ... | n - 1 |
|---|---|---|---|---|---|---|---|
| 0 | (0,0) | (0,1) | (0,2) | | (0,j) | | (o,n-1) |
| 1 | (1,0) | (1,1) | (1,2) | | (1,j) | | (1,n-1) |
| 2 | (2,0) | (2,1) | (2,2) | | (2,j) | | (2,n-1) |
| .. | | | | | | | |
| i | (i,0) | (i,1) | (i,2) | | (i,j) | | (i,n-1) |
| ... | | | | | | | |
| m - 1 | (m-1,0) | (m-1,1) | (m-1,2) | | (m-1,j) | | (m-1,n-1) |

Figure 7: $C_m(a_i) \wedge C_n(b_j)$

Two vertices in the product graph are adjacent only if their first numbers in the ordered pair are adjacent in $C_m(a_i)$ and their second numbers in the ordered pair are adjacent in $C_n(b_i)$. For example, (2,2) is adjacent to (i,j) if 2 is adjacent to i in $C_m(a_i)$ and 2 is adjacent to j in $C_n(b_i)$. Thus to determine if an edge exists between two vertices in the product graph, we need only check if an edge exists between their row headings and between their column headings. For the purpose of this paper, I will examine the case where m and n are relatively prime. Now we can number the vertices of the product graph 0 through mn-1 in the following manner, where multiplication and addition is mod mn:

| | 0 | 1 | 2 | ... | j | ... | n - 1 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | m | 2m | | jm | | (n-1)m |
| 1 | n | n+m | n+2m | | n+jm | | n+(n-1)m |
| 2 | 2n | 2n+m | 2n+2m | | 2n+jm | | 2n+(n-1)m |
| .. | | | | | | | |
| i | in | in+m | in+2m | | in+jm | | in+(n-1)m |
| ... | | | | | | | |
| m - 1 | (m-1)n | (m-1)n+m | (m-1)n+2m | | (m-1)n+jm | | 2mn-m-n |

Figure 8: $C_m(a_i) \wedge C_n(b_j)$, gcd(m,n)=1

Because m and n are relatively prime, each number in this chart is unique and thus the vertices are the set {0,1,...,mn-1}. For example, $C_3(a_i) \wedge C_5(b_j)$ is displayed in figure 9:

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 3 | 6 | 9 | 12 |
| 1 | 4 | 7 | 10 | 13 | 1 |
| 2 | 8 | 11 | 14 | 2 | 5 |

Figure 9: $C_3(a_i) \wedge C_5(b_j)$

**Theorem 5.3**: If gcd(m,n) = 1, $C_m \wedge C_n$ is an edge-transitive circulant of the form $C_{mn}$(m-n,m+n).

**Example**: $C_5 \wedge C_3 = C_{15}(2,7)$

**Proof**: It was shown in Theorem 3.1 that cycle graphs are edge-transitive, so from Lemma 5.2 we know that the cardinal product of two cycle graphs is edge-transitive. Examine the row and column headings in Figure 8. Since the circulants are cycles, each row heading will be adjacent only to the vertices immediately above and below it, and each column heading will be adjacent only to the vertices immediately to the left and right of it. Thus a vertex in the product graph will be adjacent only to the four vertices corresponding to its diagonals. For example, the vertex m+n will be adjacent to the set {0,2m,2n+2m,2n}.

On the 45° diagonal, vertices differ in number by ±(m-n). On the 135° diagonal, vertices differ by ±(m+n). Therefore if we place the vertices 0 through mn-1 on a circle, we have a circulant on mn vertices with jumps of m-n and m+n.

This gives us the first of our cardinal product families of edge-transitive graphs:

$$C_{mn}(m-n,m+n), \text{ where } \gcd(m,n) = 1 \tag{8}$$

**Theorem 5.4**: If gcd(m,n) = 1, $K_m \wedge K_n$ is an edge-transitive circulant of the form $C_{mn}(\{a \in A_{mn} \mid m \text{ doesn't divide } a \text{ and } n \text{ doesn't divide } a\})$

**Example**: $K_3 \wedge K_5 = C_{15}(1,2,4,7)$

**Proof**: It was shown in Theorem 3.2 that complete graphs are edge-transitive, so from Lemma 5.2 we know that the cardinal product of two complete graphs is edge-transitive. In this case, each row heading is adjacent to every other row heading, and each column heading is adjacent to every other column heading. Thus a vertex in the product graph $K_m \wedge K_n$ will be adjacent to everything not in its own row and column. Examine the complement: $\overline{K_m \wedge K_n}$. In the complement graph, each vertex will be adjacent to every vertex in its own row or in its own column. All vertices in the same row differ by multiples of m, and all vertices in the same row differ by multiples of n.

Therefore $\overline{K_m \wedge K_n} = C_{mn}(\{a \in A_{mn} \mid m \text{ divides a or n divides a}\})$
and so $K_m \wedge K_n = C_{mn}(\{a \in A_{mn} \mid m \text{ doesn't divide a and n doesn't divide a}\}$.

This theorem gives us the following family of edge-transitive circulants:

$$C_{mn}(\{a \in A_{mn} \mid m \text{ doesn't divide a and n doesn't divide a}\})$$
$$\text{where } \gcd(m,n) = 1 \qquad (9)$$

In this theorem we generalize the previous two families to any two edge-transitive circulants with relatively prime number of vertices.

**Theorem 5.5**: If gcd(m,n) = 1, $C_m(a_i)$ is edge-transitive and $C_n(b_j)$ is edge-transitive, then $C_m(a_i) \wedge C_n(b_j)$ is an edge-transitive circulant of the form $C_{mn}(\{a \in A_{mn} \mid a = \pm a_i n + b_j m\})$.

**Example**: $C_4(1,2) \wedge C_9(1,2) = C_{36}(1,5,10,13,14,17)$

**Proof**: Examine a sample jump a′ from $C_m(a_i)$ and b′ from $C_n(b_j)$. These two jumps will cause a vertex v to be adjacent to four vertices, those being {(v + a′n + b′m, v + a′n - b′m, v - a′n + b′m, v - a′n - b′m)}. Two of these differ from v by ±(a′n + b′m) and two differ by ±(-a′n + b′m). These two formulas hold for all possible jumps $a_i$ and $b_j$. Therefore when the vertices are placed on a circle, the graph is a circulant with all jumps of the form $\pm a_i n + b_j m$.

This gives us our final family derived by the cartesian product:

If $C_m(a_i)$ and $C_n(b_j)$ are edge transitive,
$$C_{mn}(\{a \in A_{mn} \mid a = \pm a_i n + b_j m\}) \text{ where } \gcd(m,n) = 1 \qquad (10)$$

**Note**: Families 7 and 8 can be included into 9, but since they can be more easily recognized in circulant notation, they are considered separately.

An Aside: The Cartesian Product

Another well known product of two graphs is the cartesian product. This product is not useful for proving edge-transitivity, since the cartesian product of two edge-transitive graphs is not necessarily edge-transitive, however it is interesting to examine the cartesian product of certain circulants.

**Definition 5.6**: The cartesian product of two graphs $G_1$ and $G_2$, written as $G_1 \times G_2$, is defined as follows:

The vertices of $G_1 \times G_2 = V(G_1) \times V(G_2)$.

The edges of $G_1 \times G_2 = \{((a,b),(c,d)) \mid a=c$ and $(b,d)$ is an edge in $G_2$, or $b=d$ and $(a,c)$ is an edge in $G_1$.

The following theorem is stated without proof, but can be visualized in the same manner as theorems 5.3 through 5.5.

**Theorem 5.7**: $\gcd(m,n) = 1$

a) $C_m \times C_n = C_{mn}(m,n)$

b) $K_m \times K_n = C_{mn}(\{a \in A_{mn} \mid m$ divides $a$ or $n$ divides $a\})$

c) $C_m(a_i) \times C_n(b_j) = C_{mn}(\{a \in A_{mn} \mid a = a_i n$ or $a = b_j m\})$

**Note:** $K_m \times K_n = \overline{K_m \wedge K_n}$.

**Section 6. Circulants with a prime number of vertices**

The family of edge-transitive circulants with a prime number of vertices contains many interesting patterns. Unfortunately, the logic behind these patterns and their proof remains unknown. Thus they will be stated here as speculations. Recall that all cycle and complete graphs are edge-transitive circulants, and also note that since p is prime, there are no multiple graphs in this family.

Conjecture 1) Circulants with a prime number of vertices are edge-transitive only if there exists an integer n which cyclically permutes the jump sequence (as in Theorem 3.5).

Conjecture 2) Edge-transitive circulants with a prime number of vertices have the property that the sum of the squares of the jumps is divisible by p. For example: $C_{13}(1,3,4)$ is edge-transitive with $1+9+16 = 26 \equiv 0 \pmod{13}$.

Once again, these are speculations and call for further study. However, one family of edge-transitive circulants was discovered.

**Theorem 6.1**: Given p is prime and $a^2 + b^2 \equiv 0 \pmod{p}$, $C_p(a,b)$ is edge-transitive.

**Proof**: Recall $\mathbf{Z}_p$ is a field since p is prime. $a^2 + b^2 \equiv 0$ implies $a^2 \equiv -b^2$ and $aa \equiv -bb$. So $a \equiv -a^{-1}bb$. Let $n = a^{-1}b$ which implies that $b = na$. By substituting n into the previous equation, $a \equiv -nb$. Since n cyclically permutes the jump sequence, $C_p(a,b)$ is edge-transitive by Theorem 3.5.

Thus our final family of edge-transitive circulants is:

$$C_p(a,b) \text{ where p is prime and } a^2 + b^2 \equiv 0 \pmod{p} \qquad (11)$$

## Section 7. Conclusion

We have introduced circulant graphs and their properties and have provided a proof of Adam's isomorphism theorem. We have found several families of edge-transitive circulants using tools such as the set product and cardinal product. Finally, we concluded with speculations on circulants with a prime number of vertices.

While this compilation of families of edge-transitive circulants is by no means comprehensive, it is a valuable tool not only for identifying a circulant as edge-transitive, but for understanding the nature of edge-transitive circulants.

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