Invariant Curves for Quadratic-Like Cubic Systems

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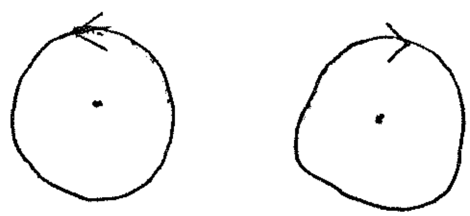
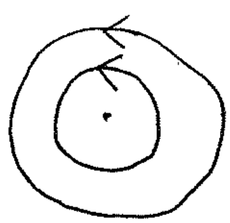
Background:

We are going to look for invariant curves for quadratic-like cubic systems. First, we need to get some background information. Schlomiuk [1] and Coppel [2] have both done work with quadratic systems. First we take a look at two theorems from Coppel's work that relate to what I am doing. Both of these come from Tung's Lemma.

Theorem 3: Two closed paths are oppositely oriented if their interiors have no common point.



Theorem 4: Two closed paths are similarly oriented if their interiors have one common point.



Another important item to notice is that 3 critical points can never be colinear in the quadratic systems. This is not necessarily true of cubic systems.

Schlomiuk's work gives me the basis to start my work. She established a connection between the existence of invariant algebraic curves in the quadratic systems and the conditions for a center. She used equations of the form:

$$x' = -y - bx^2 - Cxy - dy^2$$

$$y' = x + ax^2 + Axy + cy^2$$

The conditions for a center had already been found in the 1920's by Kapteyn[3]. Her work finding the lines and conics gives us the following precise correlation between Kapteyn's 4 center conditions and the existence of invariant algebraic curves.

$b + d = 0$	2 or 3 invariant lines
$C = a = 0$	an invariant line and conic
$0 = C + 2a = A + 3b + 5d = a^2 + bd + 2d^2$	an invariant conic and cubic
$A - 2b = (C + 2a) = 0$	an invariant cubic

Quadratic-like cubic systems are a class of cubic systems for which a modified version of Tung's Lemma hold so that for example theorems 3 and 4 stated above hold.

Vasmin[4] found conditions for the center of quadratic-like cubic systems. For the equations given at the top of page three these are:

1. $A_3 = C_3 = 0$

2. $A_2 = -4B_1$
   $C_1 = -A_1B_1$
   $C_3 = -A_3B_1$

3. $4C_3 = A_2A_3$
   $16C_2 = 8B_1A_2 - 3B_2^2 + 8B_2A_1 + 3A_2^2$
   $16C_1 = 4A_1A_2 - A_2B_2 - 4B_1B_2$

4. $4C_3 = A_2A_3$
   $16C_2 = 25A_3^2 - 40A_1A_3 + 10B_2A_3 + 8B_1A_2 - 3B_2^2 +$
   $8A_1B_2 + 3A_2^2$
   $16C_1 = 5A_2A_3 + 20B_1A_3 + 4A_1A_2 - A_2B_2 - 4B_1B_2$
   $3A_3^2 = (B_2 - 4A_1)^2 + (A_2 + 4B_1)^2 + 2A_3(B_2 - A_1)$

Using these conditions as a starting place, we start looking at quadratic-like cubic systems.

Project:

In this project, we are looking for invariant curves for quadratic-like cubic equations. We are using equations of the following form:

$$x' = y + A_1x^2 + (A_2 + 2B_1)xy + (A_3 - A_1)y^2 + x(C_1x^2 + C_2xy + (C_3 - C_1)y^2)$$
$$y' = -x + B_1x^2 + (B_2 - 2A_1)xy - B_1y^2 + y(C_1x^2 + C_2xy + (C_3 - C_1)y^2)$$

To find invariant lines, we use the general equation for a line:

$$L = sx + ly + 1 = 0$$

$$\begin{aligned} L' &= sx' + ly' \\ &= s(y + A_1x^2 + (A_2 + 2B_1)xy + (A_3 - A_1)y^2 + x(C_1x^2 + C_2xy + (C_3 - C_1)y^2) + \\ &\quad l(-x + B_1x^2 + (B_2 - 2A_1)xy + B_1y^2 + y(C_1x^2 + C_2xy + (C_3 - C_1)y^2) \\ &= sy + sA_1x^2 + s(A_2 + 2B_1)xy + s(A_3 - A_1)y^2 + sx(C_1x^2 + C_2xy + (C_3 - C_1)y^2) \\ &\quad -lx + lB_1x^2 + l(B_2 - 2A_1)xy + lB_1y^2 + ly(C_1x^2 + C_2xy + (C_3 - C_1)y^2) \\ &= sy - lx + (sA_1 + lB_1)x^2 + [s(A_2+2B_1) + l(B_2 - 2A_1)]xy + [s(A_3 - A_1) - lB_1]y^2 + \\ &\quad (sx + ly)(C_1x^2 + C_2xy + (C_3 - C_1)y^2) \end{aligned}$$

If this is 0 on L, it must factor to

$$(sx + ly + 1)\,(ax + by + dx^2 + fxy + gy^2)$$

or

$$\begin{aligned} &ax + (sa + d)\,x^2 + sdx^3 + by + (lb + g)y^2 + lgy^3 + (sb + la + f)xy + (sf + ld)x^2y + \\ &\quad (sg + lf)xy^2 \end{aligned}$$

By setting coefficients for similar terms equal, we get the following equations:

$$\begin{aligned} &a = -l \\ &sa + d = sA_1 + lB_1 \\ &sd = sC_1 \\ &b = s \\ &lb + g = s(A_3 - A_1) - lB_1 \\ &lg = l(C_3 - C_1) \end{aligned}$$

$$sb + la + f = s(A_2 + 2B_1) + l(B_2 - 2A_1)$$
$$sf + ld = sC_2 + lC_1$$
$$sg + lf = s(C_3 - C_1) + lC_2$$

From these equations, it is obvious that $a = -l$ and $b = s$. Making these substitutions, we get the following set of seven equations.

1) $-sl + d = sA_1 + lB_1$
2) $sd = sC_1$
3) $ls + g = s(A_3 - A_1) - lB_1$
4) $lg = l(C_3 - C_1)$
5) $s^2 - l^2 + f = s(A_2 + 2B_1) + l(B_2 - 2A_1)$
6) $sf + ld = sC_2 + lC_1$
7) $sg + lf = s(C_3 - C_1) + lC_2$

Once we get to this point, we can have invariant lines horizontally, vertically, and in generic cases.

Horizontal Invariant Lines

The first way which is described is with horizontal invariant lines. In this case, $s = 0$ and $l \neq 0$. With some simple calculations we get

$$a = -l$$
$$b = s = 0$$
$$d = C_1$$
$$f = C_2$$
$$g = -C_1$$

With these conditions, we will always have:

$$C_3 = 0$$

The remaining equations give us
1) $C_1 = lB_1$
5) $-l^2 + C_2 = l(B_2 - 2A_1)$

Equation 1 implies that either $l = C_1/B_1$ and $B_1C_1 \neq 0$ or that $B_1 = C_1 = 0$. In the first of these cases, we come up with one invariant line with the condition:

$$-C_1^2+(2A_1-B_2)B_1C_1+B_1^2C_2=0$$

However, when $B_1=C_1=0$, we come up with two possible l values from equation 5:

$$l^2+(B_2-2A_1)l-C_2=0$$

and hence we have two horizontal invariant lines.

Vertical Invariant Lines

For the case of vertical invariant lines, $l=0$ and $s\neq0$. Simplifying the equations, we end up with:

$$a=0$$
$$b=s$$
$$d=C_1$$
$$f=C_2$$
$$g=C_3-C_1$$

This leaves us with:

1) $C_1=sA_1$
3) $C_3-C_1=s(A_3-A_1)$
5) $s^2+C_2=s(A_2+2B_1)$

Let us first assume that $C_1=A_1=0$, using Equation 1. In this case, assuming that $A_3C_3\neq0$, we have $s=C_3/A_3$ from Equation 3 and thus one invariant line with the condition:

$$C_3^2-(A_2+2B_1)C_3A_3+C_2A_3^2=0$$

In a second case, we assume that $A_1C_1\neq0$. By Equation 1 we then know that $s=C_1/A_1$. This will again have one invariant line, but this time it will have two conditions:

$$A_1C_3=A_3C_1$$
$$C_1^2+A_1^2C_2-A_1A_2C_1-2A_1B_1C_1=0$$

There is also a third possibility, this time using Equation 3 and setting A_1, A_3, C_1, C_3 equal to 0. Then, Equation 5 gives us two possible values for s, and hence two invariant lines.

Generic Case

In this case, we have $sl \neq 0$. Using our original 7 equations, we get:

$$a = -l$$
$$b = s$$
2) $d = C_1$
4) $g = C_3 - C_1$
6)=7) $f = C_2$

and we are left with:

1) $-sl + C_1 = sA_1 + lB_1$
3) $sl + C_3 - C_1 = sA_3 - sA_1 - lB_1$
5) $s^2 - l^2 + C_2 = sA_2 + 2sB_1 + lB_2 - 2lA_1$

By adding Equations 1 and 3 we get $C_3 = sA_3$. Let us assume that $A_3C_3 \neq 0$. We then know that $s = C_3/A_3$ and $l = (A_3C_1 - A_1C_3)/(B_1A_3 + C_3)$. This will have one invariant line, and putting the s and l into Equation 5 we come up with the condition:

$$(B_1A_3 + C_3)^2(C_3^2 - A_2A_3C_3 - 2A_3B_1C_3 + C_2A_3^2) =$$
$$A_3^2(A_3C_1 - A_1C_3)(A_3C_1 - A_1C_3 + (B_2 - 2A_1)(B_1A_3 + C_3))$$

But if $A_3C_1 = A_1C_3$ and $B_1A_3 + C_3 = 0$ we then have $s = -B_1$ and again have one invariant line.

However, if we assume that $A_3 = C_3 = 0$, we can solve Equation 1 for l in terms of s and get:

$$l = \frac{C_1 - sA_1}{s + B_1}$$

By substituting this into Equation 5, we get:

$$s^4 - A_2s^3 + (C_2 - 3B_1^2 - 3A_1^2 + A_1B_2 - 2A_2B_1)s^2 +$$

$$(4A_1C_1 + A_1B_1B_2 - A_2B_1^2 - 2A_1^2B_1 - 2B_1^3 - B_2C_1 + 2B_1C_2)s +$$
$$(B_1^2C_2 - B_1B_2C_1 - C_1^2 + 2A_1B_1C_1) = 0$$

This is a quartic in s, so s has four roots and thus there are four invariant lines.

We can also write this in terms of l. In this case

$$s = \frac{C_1 - lB_1}{A_1 + l}$$

in which case our quartic expression is:

$$l^4 + B_2l^3 + (2A_1B_2 - 3A_1^2 - 3B_1^2 - A_2B_1 - C_2)l^2 +$$
$$(A_1^2B_2 - 2A_1^3 + 4B_1C_1 + A_2C_1 - A_1A_2B_1 - 2A_1C_2 - 2A_1B_1^2)l +$$
$$(A_1A_2C_1 + 2A_1B_1C_1 - C_1^2 - C_2A_1^2) = 0$$

Summary of Invariant Lines Findings

Horizontal

Always necessary: $C_3 = 0$

One invariant line: $B_1C_1 \neq 0$

$$-C_1^2 + (2A_1 - B_2)B_1C_1 + B_1^2C_2 = 0$$

Two invariant lines: $B_1 = C_1 = 0$

Vertical

One invariant line: $A_1 = C_1 = 0$

$$A_3C_3 \neq 0$$

$$C_3^2 - (A_2 + 2B_1)C_3A_3 + C_2A_3^2 = 0$$

OR

$$A_1C_1 \neq 0$$

$$A_1C_3 = A_3C_1$$

$$C_1^2 + A_1^2C_2 - A_1A_2C_1 - 2A_1B_1C_1 = 0$$

Two invariant lines: $A_1 = A_3 = C_1 = C_3 = 0$

Generic

One invariant line: $A_3C_3 \neq 0$

$$(B_1A_3 + C_3)^2(C_3^2 - A_2A_3C_3 - 2A_3B_1C_3 + C_2A_3^2) =$$
$$A_3^2(A_3C_1 - A_1C_3)(A_3C_1 - A_1C_3 + (B_2 - 2A_1)(B_1A_3 + C_3))$$

OR

$$A_3C_1 = A_1C_3$$

$$B_1A_3 + C_3 = 0$$

Four invariant lines: $A_3 = C_3 = 0$

Horizontal, Vertical, and Generic Invariant Line Combinations

In this we are looking at the various combinations of horizontal, vertical and generic line combinations. There can either be one or two horizontals, one or two verticals, and one to four generics. We will look at combinations of up to eight invariant lines.

2 Horizontal, 2 Vertical, ? Generics

First, let us look at the case of two horizontals, two verticals, and some generics.

$$2H: B_1 = C_1 = C_3 = 0$$
$$2V: A_1 = A_3 = C_1 = C_3 = 0$$

So $A_2, B_2, C_2 \neq 0$. Our equations then become:

$$x' = y + A_2xy + C_2x^2y = \quad y(1 + A_2x + C_2x^2)$$
$$y' = -x + B_2xy + C_2xy^2 = \quad x(-1 + B_2y + C_2y^2)$$

With some work it can be shown that there can be no generic invariant lines. So there are only four invariant lines.

2 Horizontal, 1 Vertical, ? Generics

$$2H: B_1 = C_1 = C_3 = 0$$
$$1V: A_1 = C_1 = 0\,;\, A_3C_3 \neq 0 \quad \text{XContradiction}$$
OR
$$A_1C_1 \neq 0 \qquad \text{XContradiction}$$

So it is impossible to have 2 horizontals and one vertical, with any number of generics.

1 Horizontal, 2 Vertical, ? Generics

$$2V: A_1 = A_3 = C_1 = C_3 = 0$$
$$1H: l = C_1/B_1$$

However, $C_1 = 0 \Rightarrow l = 0$ so l is not a line. So it is also impossible to have 1 horizontal and two verticals, with any number of generics.

1 Horizontal, ? Generics

$$1H: C_3 = 0; \; B_1C_1 \neq 0$$
$$-C_1^2 + (2A_1 - B_2)B_1C_1 + B_1^2C_2 = 0$$
$$4G: A_3 = C_3 = 0$$

With the condition, the constant term in the quartic for s cancels out, so an s can be factored out. Since we've already taken care of when s is 0, there are three possibilities for the s value. So we can have up to 4 invariant lines.

1 Vertical, ? Generics

$$4G: A_3 = C_3 = 0$$
$$1V: A_3C_3 \neq 0 \quad \text{XContradiction}$$
OR
$$A_1C_1 \neq 0$$
$$C_1^2 + A_1^2C_2 - A_1A_2C_1 - 2A_1B_1C_1 = 0$$

Again, with the condition, the constant term in the quartic for l cancels out, so an l can be factored out. Since we've already taken care of when l is 0, there are three possibilities for the l value. So once again, we can up to 4 invariant lines.

2 Vertical, ? Generics

$$2V: A_1 = A_3 = C_1 = C_3 = 0$$
$$4G: A_3 = C_3 = 0$$

$$l = (C_1 - sA_1)/(s + B_1) \Rightarrow 0 \text{ in vertical case}$$

So, it is impossible to have vertical lines and generics with no horizontals.

2 Horizontal, ? Generics

$$2H: B_1 = C_1 = C_3 = 0$$
$$4G: A_3 = C_3 = 0$$

$$s^2 - A_2 s + C_2 - 3A_1{}^2 + A_1 B_2 = 0$$

l will have one value, but s can have two values for up to four invariant lines.

1 Horizontal, 1 Vertical, ? Generics

1H: $C_3 = 0 \quad -C_1{}^2 + (2A_1 - B_2)B_1 C_1 + B_1{}^2 C_2 = 0$
1V: $A_3 C_3 \neq 0$ XContradiction
OR
$A_1 C_1 \neq 0 \Rightarrow A_3 = 0$ by Equations 1) and 3)
$C_1{}^2 + A_1{}^2 C_2 - A_1 A_2 C_1 - 2A_1 B_1 C_1 = 0$
4G: $A_3 = C_3 = 0$

We know at least one of the four generics will cancel out, because of the two conditions the constant in the quartic term in either the s or l will cancel out. But can there be five invariant lines? No.

Theorem
Yasmin's Center Condition 1. ($A_3 = C_3 = 0$) corresponds precisely to the condition that a quadratic-like cubic system has 4 invariant lines.

Proof:
$A_3 = C_3 = 0$ gives four generics unless the quadratic expressions for s or l (p.7) degenerate. This happens for example if $A_1 A_2 C_1 + 2A_1 B_1 C_1 - C_1{}^2 - C_2 A_1{}^2 = 0$, in which case there are only 3 generics. However, this condition is precisely the condition for a horizontal invariant line give at the top of p.5. Other cases are similar.

We remark that Kooij has shown that any cubic system with 4 invariant lines is integrable[5], and so it is no surprise that 4 invariant lines imply a center for the system under consideration here.

Invariant Conics for Quadratic-Like Cubics

Now, with the same equations as before, we are looking for invariant conics. The form of the conic will be:

$$D = px + qy + rx^2 + ßxy + vy^2 + 1 = 0$$

$$\begin{aligned} D' &= (p + 2rx + ßy)x' + (q + ßx + 2vy)y' \\ &= (p + 2rx + ßy)(y + A_1x^2 + (A_2 + 2B_1)xy + (A_3 - A_1)y^2 + C_1x^2 + C_2x^2y + \\ &\quad (C_3 - C_1)xy^2) + (q + ßx + 2vy)(-x + B_1x^2 + (B_2 - 2A_1)xy - B_1y^2 + C_1x^2y + \\ &\quad C_2xy^2 + (C_3 - C_1)y^3) \end{aligned}$$

If this is going to be 0 it must factor to:

$$(px + qy + rx^2 + ßxy + vy^2 + 1)(åx + \Psi y + \Phi x^2 + \partial xy + \mu y^2)$$

When we set equivalent terms equal, we end up with 14 equations.

1. $å = -q$
2. $\Psi = p$
3. $p\Psi + qå + \partial = p(A_2 + 2B_1) + 2r + q(B_2 - 2A_1) - 2v$
4. $p\partial + q\Phi + r\Psi + ßå = pC_2 + 2r(A_2 + 2B_1) + ßA_1 + qC_1 + ß(B_2 - 2A_1) + 2vB_1$
5. $p\mu + q\partial + ß\Psi + vå = p(C_3 - C_1) + 2r(A_3 - A_1) + ß(A_2 + 2B_1) + qC_2 - ßB_1 + 2v(B_2 - 2A_1)$
6. $på + \Phi = pA_1 + qB_1 - ß$
7. $q\Psi + \mu = p(A_3 - A_1) + ß - qB_1$
8. $p\Phi + rå = pC_1 + 2rA_1 + ßB_1$
9. $q\mu + v\Psi = ß(A_3 - A_1) + q(C_3 - C_1) - 2vB_1$
10. $r\Phi = 2rC_1$
11. $v\mu = 2v(C_3 - C_1)$
12. $r\partial + ß\Phi = 2rC_2 + 2ßC_1$
13. $r\mu + ß\partial + v\Phi = 2r(C_3 - C_1) + 2ßC_2 + 2vC_1$
14. $ß\mu + v\partial = 2ß(C_3 - C_1) + 2vC_2$

Simplifying some equations gives us the following results:

1. $å = -q$

2. $\Psi = p$
10. $\Phi = 2C_1$ or $r = 0$
11. $\mu = 2(C_3 - C_1)$ or $v = 0$

This gives us four cases to work with.

Case I: $\Phi = 2C_1; \mu = 2(C_3 - C_1)$
Case II: $r = 0; \mu = 2(C_3 - C_1)$
Case III: $\Phi = 2C_1; v = 0$
Case IV: $r = v = 0$.

Case I:
First assume $\Phi = 2C_1; \mu = 2(C_3 - C_1)$.

We can next find ∂ with some substitions and we get:

10. $\Phi = 2C_1$
11. $\mu = 2(C_3 - C_1)$
13. $\partial = 2C_2$

So, our cofactor coefficients are:

1. $å = -q$
2. $\Psi = p$
10. $\Phi = 2C_1$
11. $\mu = 2(C_3 - C_1)$
13. $\partial = 2C_2$

Using equations 6 and 7, we can find p:

7. $p = \frac{2C_3}{A_3}$

If we now simplify using the coefficients of the cofactors, we get the following results. Notice that equations 12 and 14 are not included because they reduce to $0 = 0$.

3. $p^2 - q^2 + 2C_2 = pA_2 + 2B_1p + 2r + qB_2 - 2A_1q - 2v$
4. $C_2p + C_1q + rp - \beta q = 2rA_2 + 4B_1r - \beta A_1 + \beta B_2 + 2vB_1$

5. $C_3p - C_1p + C_2q + \beta p - vq = 2rA_3 - 2rA_1 + \beta A_2 + B_1\beta + 2vB_2 - 4vA_1$

6. $-pq + 2C_1 = pA_1 + qB_1 - \beta$

8. $pC_1 - rq = 2rA_1 + \beta B_1$

9. $C_3q - C_1q + vp = \beta(A_3 - A_1) - 2vB_1$

Now that we have out equations set up, we are going to look for combinations of invariant lines and conics. There are many possibilities that we could have for combinations of lines and a conic. For the cases shown impossible by contradiction, I am only showing the contradictory conditions.

1 Horizontal, 1 Vertical, and a Conic

Horizontal: $B_1C_1 \neq 0$

Vertical: $A_1 = C_1 = 0$

So we have a contradiction, and therefore cannot have a conic with a horizontal and vertical invariant line in this case.

OR

Horizontal: $C_3 = 0$

$B_1C_1 \neq 0$

Vertical: $A_1C_1 \neq 0$

$A_1C_3 = A_3C_1 \Rightarrow 0 = A_3C_1 \Rightarrow 0 = A_3$

Since $A_3 = C_3 = 0$, we have four invariant lines.

1 Horizontal, 1 Generic, and a Conic

Horizontal: $C_3 = 0$

$B_1C_1 \neq 0$

Generic: $A_3C_3 \neq 0$

Again, we have a contradiction so we cannot have a horizontal and generic line with a conic.

OR

Horizontal: $C_3 = 0$

$B_1C_1 \neq 0$

Generic: $A_1C_3 = A_3C_1 \Rightarrow 0 = A_3C_1 \Rightarrow 0 = A_3$

Since $A_3 = C_3 = 0$ we again have four invariant lines.

2 Verticals and a Conic

Vertical: $A_1 = C_1 = 0$

Vertical: $A_1C_1 \neq 0$

Obviously, we have a contradiction in this case.

OR

Vertical: $A_1 = A_3 = C_1 = C_3 = 0$

Here, we have four invariant lines. So we cannot have two vertical lines and a conic.

2 Horizontals and a Conic

The condition for two horizontal lines is:

$$B_1 = C_1 = C_3 = 0$$

Since $p = 2C_3/A_3$, this implies $p = 0$.

By simplifying the remaining six equations for the conics, we get:

3. $-q^2 + 2C_2 = 2r + qB_2 - 2A_1q - 2v$
4. $-ßq = 2rA_2 - ßA_1 + ßB_2$
5. $C_2q - vq = 2rA_3 - 2rA_1 + ßA_2 + 2vB_2 - 4vA_1$
6. $0 = -ß$
8. $-rq = 2rA_1$
9. $0 = 0$

So Equation 9. will reduce to $0 = 0$, given that $\beta = 0$. We also have from Equation 8 that $q = -2A_1$ or that $r = 0$. However, if $r = 0$ there are no x terms and we are back to invariant lines. When we make these substitions into the remaining equations, we have:

3	$-2A_1^2 + C_2 = r - A_1B_2 + 2A_1^2 - v$
4	$0 = 2rA_2$
5	$-A_1C_2 + A_1v = rA_3 - rA_1 + vB_2 - 2vA_1$

Equation 4 implies that either $r = 0$ or $A_2 = 0$. Since if $r \neq 0$, we must conclude that $A_2 = 0$.

So, in summary:

$$p = 0$$
$$q = -2A_1$$
$$r = \frac{(3A_1 - B_2)(4A_1^2 - A_1B_2 - C_2) - A_1C_2}{A_3 - 4A_1 + B_2}$$
$$\beta = 0$$
$$v = r + 4A_1^2 - A_1B_2 - C_2$$

$$A_2 = B_1 = C_1 = C_3 = 0$$

1 Vertical, 1 Generic and a Conic

Case i:

Vertical: $A_1 = C_1 = 0$

$A_3C_3 \neq 0$

$C_3^2 - (A_2 + 2B_1)C_3A_3 + C_2A_3^2 = 0$

Generic: $A_3C_1 = A_1C_3$

$B_1A_3 + C_3 = 0$

So $C_3 = -A_3B_1$ by the second generic condition.
Putting that into the third vertical condition, we get:

$$A_3^2B_1^2 + A_2A_3^2B_1 + 2A_3^2B_1^2 + A_3^2C_2 = 0$$

Since $A_3 \neq 0$, we come up with

$$C_2 = -3B_1^2 - A_2B_1$$

$$p = \frac{2C_3}{A_3} = \frac{-2A_3B_1}{A_3} = -2B_1$$

3. $-q^2 - B_2q - 2r + 2v = -2B_1^2$
4. $(6B_1 + 2A_2)r + 2B_1v + B_2\beta + q\beta = 2A_2B_1^2 + 6B_1^3$
5. $(A_2B_1 + 3B_1^2)q + (3B_1 + A_2)\beta + 2A_3r + 2B_2v + vq = 2B_1^2A_3$
6. $2B_1q = B_1q - \beta \Rightarrow \beta = -B_1q$
8. $-rq = -B_1^2q$
9. $A_3B_1q = A_3\beta \Rightarrow \beta = B_1q$

So $r = B_1^2$ or $q = 0$.

Assume $1 = 0$ so that $\beta = 0$ also.

3. $v = r - B_1^2$

which gives

4. $(4B_1 + A_2)(r - B_1^2) = 0$
5. $(A_3 + B_1)(r - B_1^2) = 0$

Now if $r = B_1^2$ we have $q = \beta = v = 0$ and the invariant conic reduces to two vertical invariant lines. But if $4B_1 + A_2 = A_3 + B_1 = 0$ we get a one-parameter family of invariant conics.

$$1 - 2B_1x + rx^2 + (r - B_1^2)y^2 = 0$$

Summarzing, this occurs when

$$B_1A_3 + C_3 = 0$$
$$A_1 = C_1 = 0$$
$$A_2 + 4B_1 = 0$$
$$A_1 + A_3 = 0$$
$$C_2 = -3B_1^2 - A_2B_1 = B_1^2$$

Now, if $q \neq 0$, $r = B_1{}^2$ we have

8. $-B_1{}^2 = -B_1{}^2 q$
3. $2v = q(q+B_2)$
4. $2v = q(q+B_2)$
5. $v(q+2B_2) = 0$

So either $q = -B_2$, $v = 0$, $ß = B_1B_2$ which gives for the conic

$$l - 2B_1x + B_1{}^2x^2 + B_1B_2xy - B_2y = (l - B_1x - B_2y)$$

which is two invariant lines.

Or $q = -2B_2$, $v = B_2{}^2$, $ß = B_1B_2$ which gives for the conic

$$l - 2B_1x - 2B_2y + B_1{}^2x^2 + 2B_1B_2xy + B_2{}^2 = (l - B_1x - B_2y)^2$$

which is a double invariant line.

Case ii:

Vertical: $A_1 = C_1 = 0$

$A_3C_3 \neq 0$

$C_3{}^2 - (A_2 + 2B_1)C_3A_3 + C_2A_3{}^2 = 0$

Generic: $A_3C_3 \neq 0$

$$(B_1A_3 + C_3)^2(C_3{}^2 - A_2A_3C_3 - 2A_3B_1C_3 + C_2A_3{}^2) =$$
$$A_3{}^2(A_3C_1 - A_1C_3)(A_3C_1 - A_1C_3 + (B_2 - 2A_1)(B_1A_3 + C_3))$$

Because of the vertical condition, the left side of the generic condition is 0. Because $A_1 = C_1 = 0$, the left side of the generic condition is 0.

3 $\quad p^2 - q^2 - (A_2 + 2B_1)p - 2r - qB_2 + 2v = -2C_2$

4 $\quad C_2p + rp - ßq - (2A_2 + 4B_1)r - ßB_2 - 2vB_1 = 0$

5 $\quad C_3p + C_2q + ßp - vq - 2rA_3 - ßA_2 - B_1ß - 2vB_2 = 0$

6 $\quad -pq = qB_1 - ß$

8 $\quad -rq = ßB_1$

9. $\quad C_3q + vp = ßA_3 - 2vB_1$

6 $\quad ß = B_1q + pq$

8 $\quad r = -B_1{}^2 - B_1p$

$p=2C_3/A_3$

When plugging in everything, can solve for q in terms of v, and then wind up with two conditions.

$$A_3B_1+2A_2B_2+3B_1B_2=0$$

Case iii:

Vertical: $A_1C_1\neq 0$

$A_1C_3=A_3C_1$

$C_1^2+A_1^2C_2-A_1A_2C_1-2A_1B_1C_1=0$

Generic: $A_3C_3\neq 0$

$$(B_1A_3+C_3)^2(C_3^2-A_2A_3C_3-2A_3B_1C_3+C_2A_3^2)=$$
$$A_3^2(A_3C_1-A_1C_3)(A_3C_1-A_1C_3+(B_2-2A_1)(B_1A_3+C_3))$$

The second generic condition will reduce to 0 = 0.

6. $\beta=\dfrac{(2C_3+A_3B_1)q}{A_3}$

9. $v=\dfrac{(A_3-A_1)q}{2}$

8. $r=\dfrac{2A_1C_3^2-A_3B_1(2C_3+A_3B_1)q}{A_3^2(2A_1+q)}$

Using these and substituting into 3, 4 or 5, we get a cubic expression in q. Once we know the coefficient values, we can determine q and get more conditions.

Case iv:

Vertical: $A_1C_1\neq 0$

$A_1C_3=A_3C_1$

$C_1^2+A_1^2C_2-A_1A_2C_1-2A_1B_1C_1=0$

Generic: $A_3C_1=A_1C_3$

$B_1A_3+C_3=0$

$C_3 = -A_3B_1$

$-A_3B_1C_1 = -A_3C_1 \Rightarrow A = 0$ or $C_1 = -A_1B_1$

Assume $C_1 = -A_1B_1$. Then $p = -2B_1$

3 $q^2 + (B_2 - 2A_1)q + 2r - 2v = 8B_1^2 + 2C_2 + 2A_2B_1$

4 $A_1B_1q + (6B_1 + 2A_2)r + (B_2 - A_1)\beta + 2B_1v - q\beta = -2B_1C_2$

5 $(2A_3 - 2A_1)r + (A_2 + 3B_1)\beta + (2B_2 - 4A_1)v + vq = 2A_3B_1^2 - 2A_1B_1^2$

6 $B_1q = \beta$

8 $2A_1r + B_1\beta + rq = 2B_1^2A_1$

9 $-B_1q = \beta$

Since $\beta = B_1q = -B_1q \Rightarrow B_1 = 0$ or $q = 0$. But $B_1 = 0 \Rightarrow C_1 = 0$ so $B_1 \neq 0$.

So assume $q = 0$.

3 $r - v = 4B_1^2 + C_2 + A_2B_1$

4 $(3B_1 + A_2)r + B_1v = -2B_1C_2$

5 $(A_3 - A_1)r + (B_2 - 2A_1)v = A_3B_1^2 - AB_1^2$

8 $r = B_1^2$

Using equation 5, we can solve for v

5 $v = 1/(B_2 - 2A_1)$

Equations 3 + 4 gives us that $A_2 = 0$.

So, $p = -2B_1$

$q = 0$

$r = B_1^2$

$\beta = 0$

$v = 1/(B_2 - 2A_1)$

$A_2 = 0$

Case II:

Now let us assume that $r=0, v\neq 0$. Then we have the following equations:

3. $p^2 - q^2 + \partial = p(A_2 + 2B_1) + (B_2 - 2A_1)q - 2v$
4. $p\partial + q\Phi - ßq = C_2p + C_1q + ß(B_2 - A_1) + 2B_1v$
5. $(C_3 - C_1)P + q\partial + ßp - vq = ß(A_2 + B_1) + C_2q + 2(B_2 - 2A_1)v$
6. $-qp + \Phi = A_1p + B_1q - ß$
7. $qp + 2(C_3 - C_1) = (A_3 - A_1)p + ß - B_1q$
8. $p\Phi = C_1p + ßB_1$
9. $(C_3 - C_1)q + vp = ß(A_3 - A_1) - 2vB_1$
12. $ß\Phi = 2ßC_1$
13. $ß\partial + v\Phi = 2ßC_2 + 2vC_1$
14. $v\partial = 2vC_2$

Equation 14 implies $\partial = 2C_2$ since $v\neq 0$. We then know

13. $\Phi = 2C_1$

Since the coefficients of the cofactor are the same, this is a subcase of Case I.

Case III:

Now we assume that $v=0, r\neq 0$ and $g=2C_1$. We now have:

3. $p^2 - q^2 + \partial = p(A_2 + 2B_1) + 2r + q(B_2 - 2A_1)$
4. $p\partial + C_1q + rp - ßq = pC_2 + 2r(A_2 + 2B_1) + ßA_1 + ß(B_2 - 2A_1)$
5. $p\mu + q\partial + ßp + vå = p(C_3 - C_1) + 2r(A_3 - A_1) + ß(A_2 + B_1) + qC_2$
6. $-pq + 2C_1 = pA_1 + qB_1 - ß$
7. $pq + \mu = p(A_3 - A_1) + ß - qB_1$
8. $C_1p + rq = 2rA_1 + ßB_1$
9. $q\mu = ß(A_3 - A_1) + q(C_3 - C_1)$
12. $\partial = 2C_2$
13. $\mu = 2(C_3 - C_1)$

Once again, the cofactor coefficients are the same, which puts us in another subcase of Case I.

Case IV:

This time, we assume that both r and v are zero.

3. $p\Psi + q\aa + \partial = p(A_2 + 2B_1) + q(B_2 - 2A_1)$
4. $p\partial + q\Phi + \beta\aa = pC_2 + \beta A_1 + qC_1 + \beta(B_2 - 2A_1)$
5. $p\mu + q\partial + \beta\Psi + v\aa = p(C_3 - C_1) + \beta(A_2 + 2B_1) + qC_2 - \beta B_1$
6. $p\aa + \Phi = pA_1 + qB_1 - \beta$
7. $q\Psi + \mu = p(A_3 - A_1) + \beta - qB_1$
8. $p\Phi = pC_1 + \beta B_1$
9. $q\mu = \beta(A_3 - A_1) + q(C_3 - C_1)$
10. $0 = 0$
11. $0 = 0$
12. $\beta\Phi = 2\beta C_1$
13. $\beta\partial = 2\beta C_2$
14. $\beta\mu = 2\beta(C_3 - C_1)$

Since $\beta \neq 0$ or it would be lines, we know that $\Phi = 2C_1$ and $\partial = 2C_2$. So Case IV is also a subcase of Case I.

So in summary, we know that the cofactor coefficients are always:

$$\aa = -q$$
$$\Psi = p$$
$$\Phi = 2C_1$$
$$\mu = 2(C_3 - C_1)$$
$$\partial = 2C_2$$

We cannot have a horizontal, vertical and a conic; a horizontal, generic and a conic; or two verticals and a conic. However, we can have a horizontal and a conic or a vertical, generic and a conic.

Something interesting to note with the cofactors is that corresponding x and y terms are similar, and for the quadratic terms they are doubled. With the cubic they are tripled.

term	line	conic	cubic
x	$a=-l$	$\mathring{a}=-q$	
y	$b=s$	$\Psi=p$	
x^2	$d=C_1$	$\Phi=2C_1$	$3C_1$
xy	$f=C_2$	$\partial=2C_2$	$3C_2$
y^2	$g=2(C_3-C_1)$	$\mu=2(C_3-C_1)$	$3(C_3-C_1)$

So we know a little more about when invariant lines and conics happen. I have begun work on the invariant cubics, but haven't gotten far enough to include them in this paper. More work can be done to figure out exactly when these happen, and which cases they correspond to of Schlmiuk's work.

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