Demolition Diagrams: Using Car Crashes to test for Asphericity

Travis Kelm*
University of Oregon
trkelm@gladstone.uoregon.edu

May 7, 1996

Abstract

Introduced will be several topological notions such as relative diagrams and asphericity. Then using these tools and a recent result of Anton Klyachko, a test for the aforementioned asphericity will be described.

§1 Introduction (Two's Company)

The fact that many topologists refer to the three-dimensional sphere as the *2-sphere*, may seem a bit counterintuitive to the novice. One may wonder, it doesn't live in two-dimensionas, so why is it called the 2-sphere? The reason is that the three-dimensional sphere is formed by identifying the edges of the *two*-dimensional disk. However, the exciting fact is that the connection between the three-dimensional sphere and the number two is even richer. For example, it has long been known that the 2-sphere has Euler characteristic two, and as Stalling has shown, under the appropriate conditions, any decomposition of the three-dimensional sphere must have two "consistent items" [Sta87]. In a recent paper [Kly93], Anton A. Klyachko has further developed this relationship, establishing another wonderful connection between number two and the 2-sphere. Appropriately, his result will be proven in the *second* section.

The major result of this paper is a description of a test for asphericity which utilizes Klyachko's result. It borrows heavily from the ideas proposed in [RH95].

Because the paper is intended for (not to mention written by) the cub-mathematician a majority of the paper is spent presenting the necessary background. Sections 3 through 6 all present familiar material and can be skimmed (or even skipped) by the experienced topologist. Section 7 describes the actual test for asphericity and section 8 provides an example of the test's application, using the familiar torus (a notoriously aspherical structure) as the test case.

*This paper is the result of research done at a NSF Research Experience for Undergraduates with the help of advisor Prof. Günther Huck. Enormous thanks should also go to Kevin Kenan who made all of the figures possible, and Prof. Allan Sieradski for his expert advice.

§2 Klyachko's Lemma

Consider any finite connected graph on the 2-sphere which decomposes the sphere into a finite number of regions. Around the boundary of each of these regions let a car drive according to the following conditions:

- The i^{th} car moves only along the boundary of the i^{th} region, and never leaves it.
- The motion is continuous.
- Each point on the graph is driven over an infinite number of times. That is the cars never stop. nor do they slow exponentially.
- The cars are oriented identically and always move in the positive orientation.

The major result of Klyachko's paper is that given such car movement, we must have collisions. We shall now define precisely what we consider a "collision" and then give Klyachko's Lemma.

2.1 Definition

k-value: A point is said to have *value* k if it is the boundary of exactly k regions. That is exactly k cars pass over this point.

collision point: A point is called a *collision point* (i.e. point of collision) if more than one car occupies that point at a given moment.

total collision point: A point which has value k is a *total collision point* (i.e. point of total collision) if k cars occupy that point at a given moment.

2.2 Lemma (Klyachko's Car Lemma) *Let Γ be a connected graph on the oriented sphere, which decomposes the sphere into a finite number of regions. Let cars travel around Γ as described above. Then there exists at least two distinct points of total collision on Γ.*

Proof: First of all, we assume that no points of value 1 exist. For a point of value 1 would imply an edge with valence of only one, which implies an infinite number of total collision points.

We now argue that any *non*-total collision can be avoided in contrast to a *total* collision point, which we will show cannot be avoided, only shifted to another point.

Let s be a point of (non-total) collision. For example let the value of s be 4 and the value of the collision be 3. (See Figure 1, a.)

Notice that if we vary the speed of the cars within an ϵ neighborhood of the moment of the collision we can entirely avoid the collision. Simply speed car 1 up so that it passes vertex first. Once the "speeding" car 1 has passed the vertex, slow it down so that it will be back on its original schedule by the time which it leaves the ϵ neighborhood. Then do the same with car 2. Since car 1 has already left this edge, car 2 is safe. Finally let car 3 drive as normal. Since we only varied the speed of cars 1 and 2 within the ϵ neighborhood, we need not worry about our tinkering creating the collisions which we are endeavoring to prove.

Now if we throw in a fourth car, thus making our collision a total one, we should quickly realize that there is no way to avoid a total collision. (See Figure 1, b.)

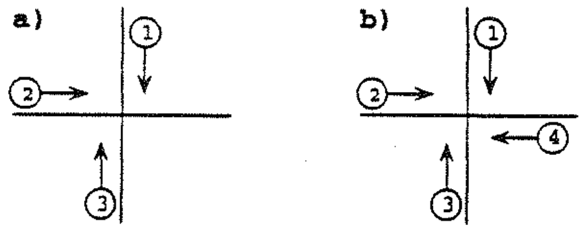

Figure 1: Collisions

Specifically, as in Figure 1b, suppose the total collision were to take place at the vertex (and hence involve all cars pictured). The same process which prevented a collision in Figure 1a, speeding car 1 up so it passes the vertex first, will only *shift* the total collision in Figure 1b. Clearly if car 1 "speeds" past the vertex first, there will be a total collision on the edge involving car 1 and car 4. Similar reasoning will show that any perturbation of the speeds of the cars will simply shift the total collision to an edge.

Now the point of this argument is that we want to guarantee that there exists a moment when *all* the cars are on an edge (equivalently, none are on a vertex). If the "natural" car schedule prevents the existence of such a moment, we have just shown that by altering the schedules of the cars within little ϵ neighborhoods of when they are on a vertex, we can *force* all the cars onto edges, while preserving the existence of any total collision which may have previously existed there. Furthermore, because we alter the schedule only within ϵ of the vertex, no new total collisions will be created. Hence, despite our tinkering, the total number of total collisions on the sphere will remain unchanged.

Thus we assume that at our starting moment, all cars are on an edge. We now define the *dual graph* as follows: (See Figure 2.)

- There is a vertex in the interior of each region.
- Each car on an edge of the original graph generates a new edge in the dual graph connecting the vertices in the two regions which the car borders. This edge is oriented, we'll say pointing out of the region which the car belongs to.

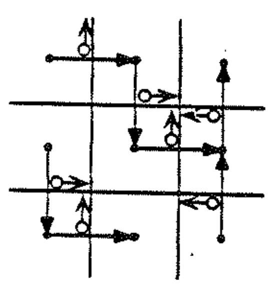

Figure 2: The car dual graph

Notice that the dual graph need not be connected, but it must have the following critical property.

2.3 Lemma *The dual graph always contains a subgraph which is a closed circle.*

Proof: Start at any vertex of the dual graph. By construction, every vertex is the origin of a dual edge. Follow the edge which originates from the starting vertex into the next vertex. Then follow the edge from this second vertex into the third. If one continues this procedure, because of the finiteness of the number of regions (vertices), one must eventually arrive at one of the vertices which has already been visited. Therefore there must exist a circle. △

Notice that the dual graph is constantly changing. That is, each edge of the dual graph rotates, not unlike the hands of a clock, as the cars travel from edge to edge. Furthermore by tinkering with the car schedules as we did previously, we can be certain that only *one* car changes edges at any single moment.

The implications of this are crucial for the proof. First of all, the finiteness of the number of cells and the previous lemma imply that an innermost (equivalently "outermost" if viewed from opposite side of sphere) circle exists. It is a necessary condition that a closed circle be consistently oriented. If not, then at least one vertex would have two edges radiating outward, an impossibility by definition.

Now suppose the innermost circle is rotating inward. Let λ be the dual edge on the circle which will rotate before any of the others. Certainly once λ (and only λ) has made this rotation, our circle will be destroyed. Yes this is true, but the fact is that now an even smaller inward rotating circle has been created. To show this we use an argument similar to that of Lemma 2.3. Simply consider the vertex which λ now points into, call it v_λ. Suppose the dual edge which emanates from v_λ point into $v_{\lambda+1}$. If you keep following dual edges in such a manner we will eventually run back into an edge that was part of our old inward rotating circle. Therefore what we have is a new smaller inward rotating circle. What we are observing is a *process of contraction* as the edges rotate inward. This process will continue until two dual edges overlap, each oriented oppositely. This phenomena corresponds to a total collision on one of our edges. (See Figure 3.)

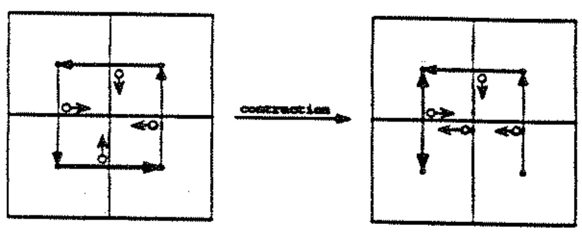

Figure 3: A process of contraction

But what if the innermost circle is rotating outward? Well in this case it must be true that if we reverse time, our circle will rotate inward. Hence if we "travel back in time" we will observe the above process of contraction, and find our total collision point in the past.

However these arguments make an assumption. Certainly it is possible for there to exist edges on the interior of our innermost circle. And it is certainly possible that these "interior edges" will, by the motion of edges of the dual graph, form a new innermost circle before our original innermost circle completes its process of contraction. Clearly if this new innermost circle rotates inward, we have no problem. However, if it rotates outward we could have trouble. We could have an innermost circle which contracts inward for a bit, until a new innermost circle forms, which contracts out for a bit, until a new one forms, which contracts in the opposite direction, and so on and so forth. In a nutshell, our process of contraction would keep "flip-flopping" and would never end. Now if we can show that this, in fact, can't happen, we will have proven that at least one point of total collision exists.

2.4 Lemma *A smaller outward rotating circle cannot form on the interior of an innermost inward rotating circle.*

Proof: At time $t = 0$ let our inward rotating circle (IIRC) be innermost, that is, let it have only non-circle edges in it's interior. At $t = 1$ let a smaller outward rotating circle (IORC) form inside the interior of IIRC. Now move back to $t = 0$. If we reverse time, any outward rotating circle must contract, forming an even smaller circle. Hence at $t = 0$ our IORC is even smaller than it is at $t = 1$, certainly smaller than IIRC. This contradicts assumption that at $t = 0$ the IIRC was the innermost circle. $\triangle$

So now we've shown that there is at least one collision on the sphere. However, Klyachko says that we have at least two, and one little topological argument is all the more we need to get it.

Suppose that there is only one point of total collision on the sphere. Now take two copies of the sphere and connect them with a tube around an ϵ neighborhood of the collision points. (See Figure 4. a.) Now we notice that by doing this a single new region is created (namely the tube itself, with edge self identifications) and hence a single new car. However we can also see that we can design a schedule, so that the collision we once had is no longer there. (See Figure 4, b.) Thus, because we assumed that only one total collision point existed on our original sphere, it is clear that this "double sphere" construction has *no* points of total collision.

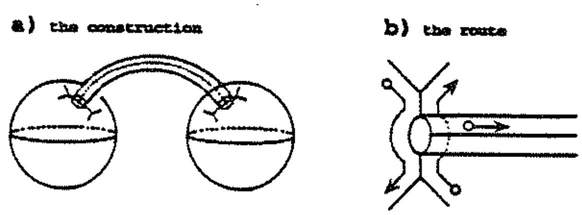

Figure 4: Topological argument

But by construction, the "double sphere" is homeomorphic (topologically equivalent) to the original sphere. Thus all the arguments which held for the original sphere must hold for the "double sphere." Namely, there exists at least one total collision point.

So we have a contradiction. Therefore the assumption that there exists *only one* point of total collision on the sphere is bad. This completes the proof of Klyachko's Lemma: any finite decomposition of the sphere must have at least two points of total collision. $\square$

§3 Combinatorial Group Theory

You all certainly know from your algebra classes that a *group* is a set along with a law of composition such that the set contains an identity element, and every element has an inverse in the set. This definition still holds in *Combinatorial Group Theory*; only the way that groups are presented changes.

In combinatorial group theory, groups are presented in terms of generators and relations.

3.1 Definition Let G be a group. Let $X \subseteq G$. X is called a *system of generators of* G if every element of G is expressible as a product of elements of X and their inverses.

3.2 Example $\{1\}$ is a generating system for the integers.

3.3 Definition A *word* is a finite string of symbols in which repetition is allowed.

So given a system of generators, X, we say that every element of group G is a word formed from elements (and their inverses) of our generating system.

3.4 Example Suppose $\{x, y\}$ is a generating system for G. $xyxxy^{-1}$, yyy, and, xx^{-1} are all valid words, and it should be clear that they each represent an element of G. Notice that each group element can be written in an infinite number of unique ways: $xy = xx^{-1}xy = xyy^{-1}y$ and so on.

3.5 Definition A cyclically reduced word which equals the identity in G, is called a *relator*. A set R of relators is called a *system of defining relators* if every relator is a consequence of those in R.

3.6 Example $1+1$ is the system of defining relators for $\mathbb{Z}_2$, because

$$\begin{aligned} 1+1=0 &\Rightarrow 1+1+1+1=0 \\ &\Rightarrow 1+1+1+1+1+1=0 \end{aligned}$$

and so on.

We now define the *Presentation* of a group G to be simply a list of generators followed by a list of the defining relations. We denote it as

$$G = < x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m >$$

Algebraically it represents the group

$$G = F(x_1, x_2, \ldots, x_n) \, / << R_1, R_2, \ldots, R_m >>$$

where $F(x_1, x_2, \ldots, x_n)$ is the set of all words formed by the elements of $\{x_1, x_2, \ldots, x_n\}$ (often called the *free group* on n elements) and $<< R_1, R_2, \ldots, R_m >>$ is the normal subgroup of G generated by the system of relations $\{R_1, R_2, \ldots, R_m\}$. In other "words", what we have is the set of all words $((F(x_1, x_2, \ldots, x_n))$ with all of the words which should equal the identity $(<< R_1, R_2, \ldots, R_m >>)$ "moded" (i.e. collapsed) to the identity..

3.7 Example

1. $< a \mid \{\} >$ is the group generated by one element with no imposed relations. It is called *the Free Group with one generator*. Think of the integers under addition.

2. $< a, b \mid aba^{-1}b^{-1} >$ is the commutative group with two generators because $aba^{-1}b^{-1} = 1 \Rightarrow ab = ba$. Think of $\mathbb{Z} \times \mathbb{Z}$ under addition.

3. Recall the *Klein four group*, the simplest group which is not cyclic. It is often thought of as the four matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \tag{1}$$

In terms of combinatorial group theory we would write

$$< a, b \mid a^2, b^2, (ab)^2 >$$

where a, b are any two of the matrices in (1) other than $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

§4 Diagrams

Before we start talking about diagrams and cell decompositions of the sphere, we first need some definitions. Our goal is to come up with a geometric representation of our group presentation, a *2-complex* modelled on our presentation. Naturally we shall begin by defining what a 2-complex is.

We begin simply enough with a finite set of discrete points [1] This is called our ***0-skeleton***, for it is composed of *0-cells*, that is cells of dimension zero.

We now take a set of *1-cells* (which are essentially closed intervals; cells with dimension 1) and attach them to our 0-skeleton so that the 1-cell boundaries are mapped onto the members of our 0-skeleton. So basically what we have now is a graph. We shall call it a 1-skeleton.

Finally we take *2-cells* (which are essentially closed disks) and attach them to our 1-skeleton in a similar manner. The result is what we will call a 2-complex.

Now we will build the *standard 2-complex* for a group presentation. The idea is this: (see Figure 5)

1. Form an oriented 1-skeleton with a loop labelled for each group generator.

2. For each defining relation draw a 2-cell whose boundary is labelled by that relation (as in Figure 5).

3. Now attach the 2-cells to the 1-skeleton as described above.

You have just defined the standard 2-complex modelled on a group presentation.

But what can we do with this complex?

The fact is that complexes are intricately related to the spherical diagrams which Klyachko's lemma work on. Hence in order for Klyachko to be of any value to us we need to understand this relationship.

4.1 Definition Let K_P be the standard 2-complex of presentation P. Let S be a cellular decomposition of the 2-sphere. A *spherical diagram* is a map $f : S \longrightarrow K_P$, such that f maps open cells of S homeomorphically onto open cells of K_P.

[1] In full generality the set need not be finite but for simplicity we shall assume finiteness.

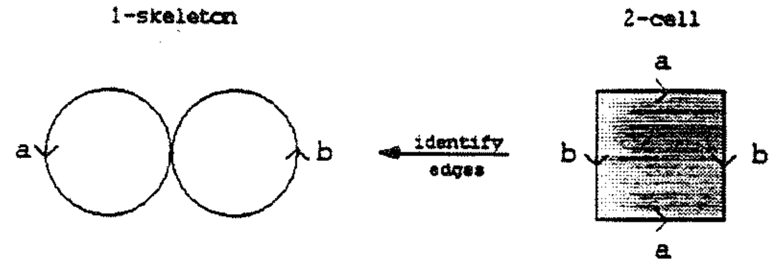

Figure 5: Forming the standard 2-complex for $< a, b \mid aba^{-1}b^{-1} >$

You're probably thinking that this term "homeomorphically" could use some explanation. It simply means that the 0-cells of S are mapped to 0-cells, the 1 cells are mapped to 1-cells, and the 2-cells are mapped to 2-cells. The idea is that the manner in which the sphere is decomposed is determined by how it will map onto the 2-complex.

An intuitive way to think about it is this. Imagine painting the 1-skeleton of K_P such that each 1-cell has its own color. Now color the 2-cells with white paint. Before this paint can dry, take a plain 2-sphere and "moosh" it down onto K_P so that every part of the sphere makes contact with some part of K_P. Now "unmoosh" the 2-sphere. The 2-sphere will now have paint all over it, specifically it will have colored paint where it made contact with the 1-skeleton and white paint where it touched one of the 2-cells.

Now if we "moosh" the newly painted sphere onto K_P just as we did before, the 0-cells of S will land on 0-cells of K_P and so on. This is a spherical diagram.

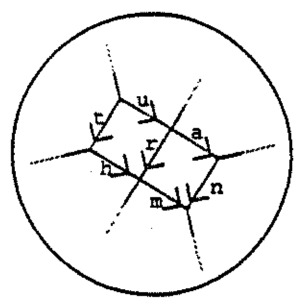

Figure 6: A spherical diagram

§5 Asphericity and DR

As the title indicates, there is more to this paper than Klyachko's lemma about car crashes on diagrams. Indeed, there is this business about "asphericity," whatever that is.

5.1 Definition We say that a presentation P, with 2-complex K_P, is *aspherical* if, every spherical diagram over K_P is homotopically trivial.[2]

[2]This is actually a "weak" definition of asphericity. Generally a presentation P, with 2-complex K_P, is called aspherical if every continuous map from the 2-sphere to K_P is homotopically trivial. Now, it is true that this general

Now the question is, what's all this business about "homo-topy."

5.2 Definition Let X and Y be spaces. Let $\mathbb{I}$ be the unit interval $[0,1]$ and $0 \leq t \leq 1$. Define i_t to be an inclusion function $i_t : X \longrightarrow X \times \mathbb{I}$ such that $i_t(x) = (x,t)$. Finally let f, g be continuous functions from $X \longrightarrow Y$. Then a *homotopy* is a continuous function $H : X \times \mathbb{I} \longrightarrow Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Notice that H is sort of a "continuous deformation" from f to g. When such an H exists, we say that f and g are *homotopic* and that H is a *homotopy* from f to g.

Hence it follows that a *homotopically trivial* map is a continuous function which is homotopic to a constant function.

5.3 Example Let $X = (0,1) \subset \mathbb{R}$ and $Y = (1,2) \subset \mathbb{R}$. The function $f : X \longrightarrow Y$ defined by $f(x) = x+1$ is homotopic to constant map $g(x) = \frac{3}{2}$ via the homotopy $H(x,t) = (1-t)(x+1)+\frac{3}{2}t$. Hence f is homotopically trivial.

Now we are very close to realizing the purpose of this paper. A bit more background and we will be ready.

Related to the notion of asphericity is the notion of *diagrammatic reducibility.*

5.4 Definition A complex is called *diagrammatically reducible* if every spherical diagram over it contains an *elementary fold.*

Now an *elementary fold* is when a 2-cell in the diagram representing relation R bumps "nicely" into a cell reading R^{-1}. More precisely, the two 2-cells have at least one edge, m, in common and the map f into K_P folds them together with the edge m acting as "the hinge". In particular the labeling of the boundaries of the two 2-cells must be symmetric across the common edge m. Hence we get a minor cancellation, and what once were two cells becomes a single line. (This change of the diagram is in fact a homotopy). But like the definition says, *every* spherical diagram contains a fold, hence the new diagram will have a fold as well, so we can keep making these folds until there is only a pair of regions left. Since we are diagrammatically reducible, these final two regions will be mirror images of one another, that is they will both map to the *same* two cell in K_P. Such a diagram is thus homotopically trivial. Therefore we have the following theorem.

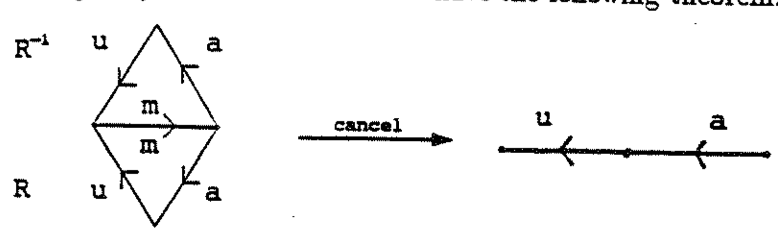

Figure 7: An elementary fold and minor cancellation

5.5 Theorem *Every diagrammatically reducible presentation is aspherical.*

definition of asphericity does indeed follow from our "weak" asphericity, but the details of the relationship are beyond the scope of this paper (but can be found in [HAMS93]). Thus rather than making wild claims and providing little proof, we have chosen to simply use this weaker definition.

5.6 Remark We will also use the notion of a "reduced" diagram which is a diagram which does *not* contain an elementary fold. In view of this notion we can say that a presentation P is diagrammatically reducible if there does not exist a *reduced* spherical diagram over P.

§6 Whitehead Graphs

While initially the notion of the Whitehead graph may seem a bit obtuse. the fundamental ideas underlying its definition are pretty straightforward. It may help to think of it as a kind of dual graph. because its form is dependent upon that of another graph. The Whitehead graph has proven a valuable tool, not only in the test for asphericity presented here. but also for the *weight test* another widely employed test for asphericity.

6.1 Definition Let K_P be the standard 2-complex of presentation P. The *Whitehead graph* W_P of K_P is the boundary of a regular neighborhood of the single vertex of K_P.

Now if you've never encountered this definition before, you are no doubt scratching your head a bit. Let's first think about precisely what the definition is saying, and then I will show you a quick and dirty way of representing the Whitehead graph of a given standard 2-complex.

Recall what a standard 2-complex is. (Look back at Figure 5.) What we have is one or more 2-cells attached to our 1-skeleton. Notice that there is only a single vertex in our standard 2-complex. namely the single 0-cell to which our generator labeled 1-cells are attached to. Now using Figure 5 as guide, visualize what happens when we attach the 2-cell to the 1-skeleton by making edge identifications. The top edge of our 2-cell will "wrap" around the left loop of the 1-skeleton, then the right edge of the 2-cell will "wrap" around the right loop, and so on, until the 2-cell is completely attached. The result of course is the standard 2-complex.

Now once the 2-cell is attached, imagine painting a small circle around the vertex of the standard 2-complex. The interior of the circle is naturally a neighborhood of the vertex, while the circle itself is the boundary of this neighborhood, i.e. the circle *is* the Whitehead graph.

I can tell you're not impressed. The Whitehead graph is still nothing more than a circle abstract enough to make Picasso cry for mercy. The neat thing, however, is what happens if we now *detach* the 2-cells from the 1-skeleton.

If one thinks carefully (and for a long time. as was the case for myself) about how we formed the standard 2-complex, one will realize that the circle painted on the standard 2-complex is simply the composition of little arcs painted around the corners of the 2-cells. The Whitehead graph of Figure 5 is simply the four $\frac{\pi}{2}$ radian arcs located in the corners of the 2-cell. (See Figure 8)

Notice that all these arcs do is form "bridges" across the corners of the 2-cell. And just as the importance of a bridge derives from what two pieces of land it links, so too can all the important information of the Whitehead graph can be represented in terms of what precisely these arc/"bridges" link together.

Specifically, consider the 2-cell in K_P which represents our particular relation. Each edge is labelled by a generator, x_i, and is oriented depending upon whether or not x_i or x_i^{-1} is read in the relation. Because of the existence of an orientation, each edge has both a beginning and an end. So what we're going to do is add two new labels to each edge, specifically we put a "$-$" at the *start* of each edge, and a "+" at the *end*. (See Figure 9)

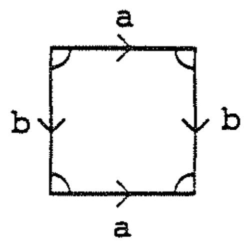

Figure 8: The Whitehead graph of the standard 2-complex from Figure 5

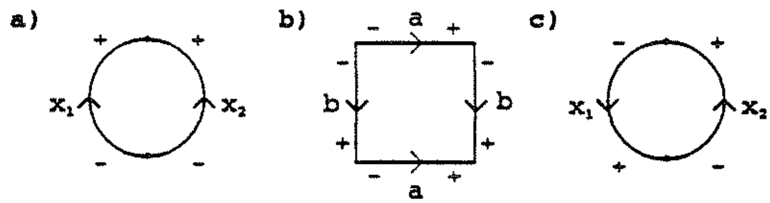

Figure 9: More labels for our 2-cells

Thus it should be clear that the Whitehead graph can be represented as a graph whose vertices are generating elements labeled with a "+" or a "−."

Now if you didn't follow any of that you can still generate a Whitehead graph quite easily by using the following algorithm.

1. The number of vertices in W_P is determined by the rank (number of elements) of our generating system $X \subseteq P$. For every $x_i \in X$ there are two vertices in W_P, normally labelled ${x_i}^+$ and ${x_i}^-$.

   6.2 Example Pretend $X = \{x, y\}$. Then our vertices in W_P would be x^+, x^-, y^+, y^-.

2. The number of edges in W_P is determined by the total number of "vertices" in our system of relations. The key is to remember how we formed our standard 2-complex. The 2-cells we used were cells whose edge borders read off members of our generating system. Naturally these cells have vertices, and these vertices are precisely what we are concerned with. The moral of the story is that for every vertex in a relation 2-cell, there is an edge in W_P.

So we've found the raw number of how many vertices and how many edges are in W_P, but we are still a distance from knowing all about W_P. I mean we still don't know the manner in which the vertices are connected via our edges. But we will shortly.

Naturally, since our W_P is a sort of "dual graph," the edges of W_P are determined by the formation of K_P. Consider vertex v_k in the 2-cell. It will be bordered by two oriented edges labelled by generators, call them x_j, x_l. With our new labels we can say that either the + or the − side of x_j, x_l touches v_k. This is precisely how we get our W_P. If the + side of x_j and the + side of x_l touch v_k, then the edge in W_P which corresponds to v_k is drawn from ${x_j}^+$ to ${x_l}^+$. (See Figure 10)

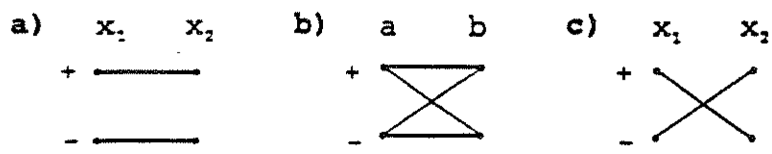

Figure 10: The Whitehead graphs for relators of Figrure 9

6.3 Remark Important for our test for asphericity is that the edges of W_P be directed. We will adopt the convention that if we read around our relation from $x_j^{\pm 1}$ to $x_l^{\pm 1}$. then our W_P edge is directed from $x_j^{\pm}$ to $x_l^{\mp}$.

§7 The Test for Asphericity

We will now explain in general, the test for asphericity which utilizes the result of Klyachko's Lemma. In order to stimulate understanding, there are several figures in this section which serve as examples of precisely how the test works. Section 8 provides a full scale application, using this test to prove that the torus is aspherical.

The general idea is this: Consider a presentation and the *reduced* spherical diagram over this presentation. Thinking of this spherical diagram as a tesselation of the 2-sphere, we endeavor to find a car schedule for this tessalation such that *no* collisions occur. This is a boldface contradiction of Klyachko's Lemma, thus it follows that such a reduced spherical diagram does not exist. In other words, any spherical diagram over our presentation is reducible—it contains an elementary fold. Thus by Definition 5.4 the presentation is diagrammatically reducible, and hence aspherical.

First we need to come up with a car schedule for the spherical diagram over our 2-complex. This can of course be anything, but for our examples we will keep it simple and let the cars drive across a single edge every unit of time.

Then, with this schedule in place, we will check for collisions on the vertices. The check I describe here requires even more labeling in our Whitehead graph, but it's not so bad. Each edge in W_P corresponds to a vertex in our relation, which we have labeled with integer times according to our car schedule. Well, now simply label each directed edge in W_P with the time corresponding to when the car arrives there on the spherical diagram. But we must also consider the relators inverse, for it is a legal cell on our spherical diagram. So on each edge in W_P label the *opposite* orientation of the edge according to the time when the vertex is reached in the inverse relation. (See Figure 11.)

Now the fact of the matter is that a collision on a vertex corresponds to a reduced closed consistently oriented path in W_P, such that the same time occurs on each edge of the path. (See Figure 12.) So we simply examine W_P, and if no such path occurs, then we know a vertex collision cannot happen.

Next we check for collisions on edges. It, like the test for vertex collisions, uses the double labels on W_P, and is quite convenient, as it tells us what we need to know simply by looking at the Whitehead graph. However, this check is slightly more complex in the sense that we need to keep track of relations (why we need to keep track will become apparent with Case 7.1). Thus what we're going to do is *color* the edges in the Whitehead graph. The edges should be colored such that edges coming from identical relations have identical colors, but edges from distinct relations have

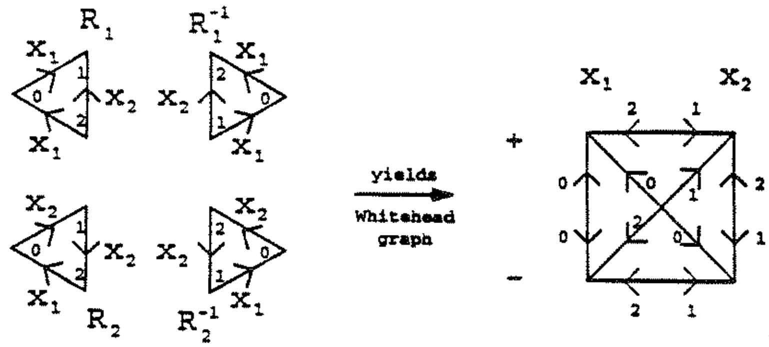

Figure 11: A labeled Whitehead graph resulting from relations R_1, R_2 and a specific schedule

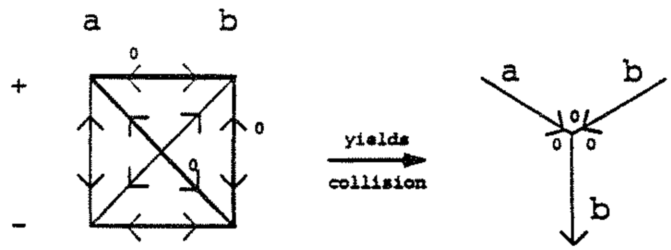

Figure 12: This path implies a vertex collision

distinct colors. In other words, simply color the edges based upon which relation generated them.

In order for there to be a collision inside an edge, we need for one car to enter one side of edge x_i at time t, while another car enters the opposite side of the same edge at same time. (See Figure 13.) Obviously this forces a collision on x_i to be inevitable. Furthermore because of the way we set our schedule, both cars will leave the edge at time $t+1$.

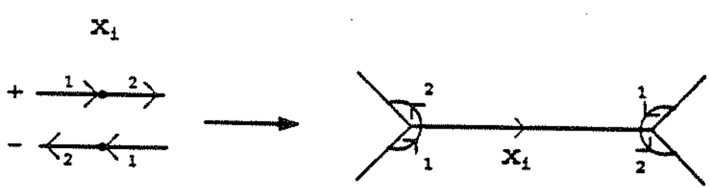

Figure 13: An edge collision

We now consider how this situation would be reflected in W_P. In order for there to be collision on edge x_i between times t and $t+1$ we need all of the following to be satisfied:

- An edge which points into x_i^- at time t.
- An edge which points into x_i^+ at time t.
- An edge which points out of x_i^- at time $t+1$.
- An edge which points out of x_i^+ at time $t+1$.

Any set of edges (with the exception of Case 7.1 below) which satisfy all the above criteria will imply an edge collision.

7.1 Case Let x be an element of relation R_0, x^+, x^- be vertices in Whitehead graph corresponding to x. Let α, β be two edges in W_P which are same color (that is, they come from the same relation) such that

1. α points into x^+ at t, and out of x^+ at $t+1$.
2. β points into x^- at t, and out of x^- at $t+1$.

This case does not imply a collision because were two regions on spherical diagram to bump into each other in the manner dictated by these edges in W_P, there would be an elementary fold (Definition 5.4 and Figure 7) Thus even though our collision criteria are satisfied by this case, we are allowed to ignore it, because we assume we are working with a *reduced* spherical diagram (so no folds can exist). Furthermore, there is an equivalence between elementary folds and Case 7.1 (See Figure 15), thus Case 7.1 is the only case we need to exclude from the edge collision test.

7.2 Example In Figure 14 we see that the relation R implies the Whitehead graph W_R.

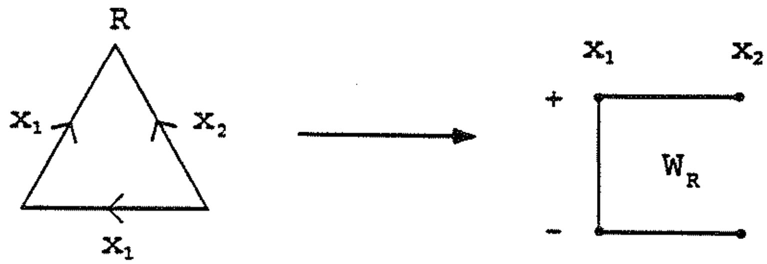

Figure 14: W_R

If we assign the schedule for cars running in R and R^{-1} as shown on the left side of Figure 15, then clearly the labeled subgraph of W_R pictured on the right side of Figure 15 contains edges which satisfies the four edge collision criteria. However, clearly the right hand side is an example of Case 7.1 and thus we claim we can ignore it, for it implies an elementary fold. Indeed, as the arrow indicates, the graph on the right hand side is implied by the adjoining of cells as shown on the left hand side, and clearly the left hand side is an example of an elementary fold.

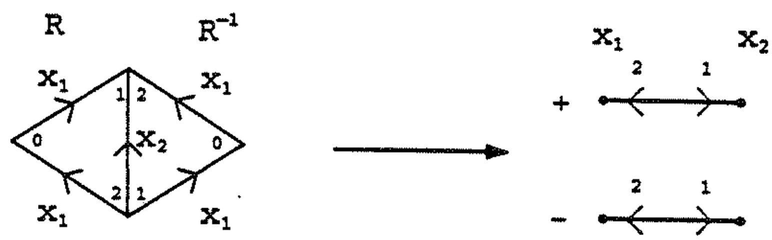

Figure 15: A trivial case

So once again, we simply analyze W_P and if we have the case above we know we are out of luck. But don't despair, true-believers, this test is merely a positive test. That is, by passing the test we are assured that our presentation is aspherical. However a failure tells us nothing, other than that collisions can occur for a spherical diagram with our particular car schedule. Indeed, the sneaky reader will have already realized that the given, simple, one edge per second, car schedule which we worked with, is by no means set in stone. The idea is that, should one car schedule fail, we should feel free to perturb the car schedule, so as to eliminate the conflicting times. In many cases this will simply shift the problem around, but in others it may lead to a schedule for which no collisions occur, hence proving asphericity.

§8 The Asphericity of the Torus

In this section we will demonstrate how wonderfully easy and effective the test for asphericity is by using it to show that the torus is aspherical.

Recall that the torus is formed by identifying the opposite edges of a disk, as in Figure 16.

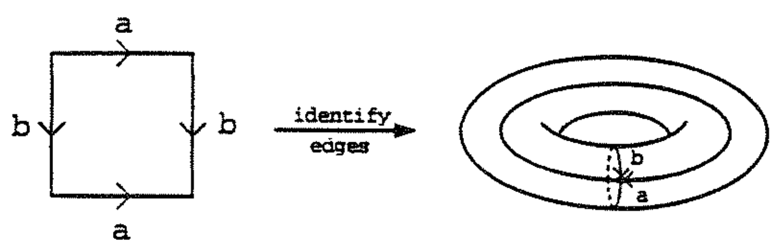

Figure 16: The torus

That is, the torus is the standard 2-complex of the group $T = \ll a, b : aba^{-1}b^{-1} \gg$. Thus the formation of the torus involves a single 2-cell, namely the one pictured in Figure 16. Therefore, every spherical diagram over the torus can be thought of as a cellular decomposition of the 2-sphere, where the boundary labels of the cells read off $aba^{-1}b^{-1}$ or $bab^{-1}a^{-1}$ (See Figure 17)

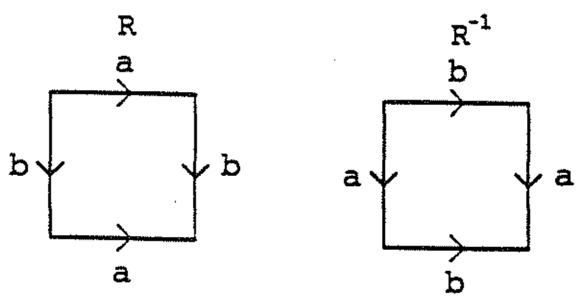

Figure 17: The cells of the spherical diagram

So if we wish to apply Klyachko's Lemma, we want to have cars driving along the edges of cells like those pictured in Figure 17. Thus, the first order of business is to determine a schedule for our cars, that is, exactly how do we want our cars to drive along these edges. For simplicity we will have the cars travel on edge per second and both "begin" in the upper left hand corner of the cells in Figure 17. (In other words. at every integer time t such that $t \equiv 0 \bmod 4$ the cars in both R and R^{-1} are at the vertex from which both an "a" edge and a "b" edge point out of.) (See Figure 18)

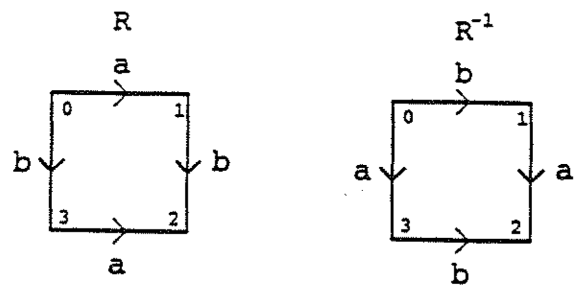

Figure 18: The car schedule

With a schedule now determined we can draw a labeled Whitehead graph for the 2-complex. It is shown in Figure 19[3].

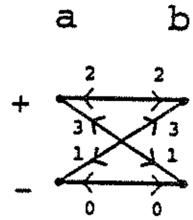

Figure 19: W_T

We can now simply examine W_T for the presence of any of the situations which would indicate a total collision. First we check for the existence of a reduced, closed, consistently oriented path which has the same time on each edge of the path, for such a path would imply that a vertex collision could occur. It should be fairly obvious that no such path exists in Figure 19.

We now check for the existence of edges described on page 13, that is

- An edge which points into x_i^- at time t.
- An edge which points into x_i^+ at time t.
- An edge which points out of x_i^- at time $t+1$.
- An edge which points out of x_i^+ at time $t+1$.

for the existence of such edges imply that edge collisions are possible. Again a quick survey of Figure 19 reveals that the only edges which satisfy the criteria also fall into Case 7.1, and thus can be disregarded. (Notice that there exists edges numbered 0 and 1 which go "into" a^- and b^-, but no edges labelled 0 or 1 go into a^+ or b^+, thus there is no way that the four criteria can be met.)

Thus we have shown that *no* total collisions of any sort can occur on a reduced spherical diagram over the torus with the aforementioned car schedule. This is a contradiction of Klyachko's Lemma,

[3] Since the presentation of the torus involves a single relation, we need not worry about having to color W_T.

which dictates that there must be at least *two* total collisions. This leads us to conclude that such a reduced spherical diagram does not exist. Therefore every spherical diagram over the torus is diagrammatically reducible. and hence the torus is aspherical.

References

[Bis94] Igor Biskup. A new property of the sphere. equations over groups and the Kervaire-Laudenbach-Conjecture. Undergraduate thesis. Johann Wolfgang Goethe University. Frankfurt, Germany. Supervised by Prof. Dr. Wolfgang Metzler, February 1994.

[CZ89] D.J. Collins and H. Zeischung. *Combinatorial Group Theory and Fundamental Groups.* I.H.E.S. Publications, 1989.

[HAMS93] Cynthia Hog-Angeloni, Wolfgang Metzler, and Allan J. Sieradski. *Two-dimensional Homotopy and Cominatorial Group Theory.* Number 197 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1993.

[Kly93] Anton A. Klyachko. A funny property of sphere and equations over groups. *Communications in Algebra*, 21(7):2555–2575, 1993.

[RH95] Stephan Rosebrock and Günther Huck. A test on asphericity and some applications. Pre-Print, 1995.

[Sie92] Allan J. Sieradski. *An Introduction to Topology and Homotopy.* PWS-KENT Publishing Company, 1992.

[Sta87] J.R. Stalling. A graph thoeretic lemma and group embeddings. *Annals of Mathematical Studies*, 111:145–155, 1987.