Bouncing Balls and Bifurcations: Examining a Discontinuous Iterated Map

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Abstract

The author studies the mathematical properties of a ball bouncing on a sinusoidally vibrating table, a system first studied by P. J. Holmes. After pointing out problems in the approximate Holmes map, an exact map for the system is derived. Many interesting phenomena are explored, including sticking solutions, glancing collisions, and discontinuities in the map.

1 Introduction

The mathematical qualities of a ball bouncing on a sinusoidally vibrating table were first examined by P. J. Holmes, where he demonstrated that the ball's motion is often times chaotic [3]. To reduce the system to one tractable to analysis, Holmes made the approximation that the distance traveled by the ball between impacts is large compared to the motion of the table. This allows one to derive an easy approximation of the time interval between impacts and the impact velocity. We let α denote the ball's coefficient of restitution; ω and β, the table's angular frequency, and amplitude, respectively; V, the ball's dimensional velocity; and g, as usual, the acceleration of gravity. Using these variables, Holmes's simplifications lead to the map

$$f = f_{\alpha,\gamma} : \begin{array}{l} \phi_{j+1} = \phi_j + v_j \\ v_{j+1} = \alpha v_j - \gamma \cos(\phi_j + v_j) \end{array} , \tag{1}$$

where $\phi = \omega t$, $v = 2\omega V/g$, and $\gamma = 2\omega^2(1+\alpha)\beta/g$.

As Holmes demonstrated, this map seems to have a strange attractor (reproduced in Figure 1), though he did not prove its existence. Looking at this

We take the same equation of motion for the table as Holmes: $-\beta \sin(\omega T)$. By differentiating this expression with respect to t, we get the table's velocity:

$$W(T_j) = -\beta\omega \cos(\omega T_j). \tag{4}$$

Also, from basic physics we know that the ball's approaching velocity is given by

$$U(T_j) = V(T_{j-1}) - g(T_j - T_{j-1}). \tag{5}$$

Substituting Equations (4) and (5) into the rearranged impact equation (3) and shifting subscripts, we get

$$V(T_{j+1}) = -\alpha V(T_j) + \alpha g(T_{j+1} - T_j) + (1+\alpha)(-\beta\omega \cos(\omega T_{j+1})). \tag{6}$$

Multiplying this equation by ω/g and collecting terms, we can cast it in the form

$$v_{j+1} = -\alpha(v_j - (t_{j+1} - t_j)) - (1+\alpha)a \cos t_j, \tag{7}$$

where $v = \omega V/g$, $t = \omega T$, and $a = \omega^2\beta/g$.

A note about why we would wish to rewrite Equation (6) in terms of the above parameters: Doing so accomplishes two things. First, all variables in Equation (7) are dimensionless; thus, in future analysis and numerical computation, we do not need to worry about what units system we are in. Second, we can now readily see that the map depends on just two parameters: α and a, so we can concentrate our efforts on understanding how these two variables affect the behavior of the system. These nondimensionalized equations might make more sense if the reader thinks of the variables in this way: v and t are dimensionless velocity and time, respectively; α is the familiar coefficient of restitution; and a is a measure of how violently the table is vibrating (more specifically, it is the table's maximum acceleration expressed as a fraction of g).[2]

Now that we have an expression for v_{j+1} in terms of v_j and t_j we seek a similar expression for T_{j+1}. To find the time of the next impact, we are looking for the next time when the ball and the table are at the same height. The table's height at any time T is given by

$$H_{table}(T) = -\beta \sin(\omega T). \tag{8}$$

From high school physics we know that the ball's height is given by

$$H_{ball}(T) = y_0 + V_0(T - T_0) - 1/2g(T - T_0)^2, \tag{9}$$

where y_0, V_0, and T_0 are the initial height, velocity and time, respectively. To determine T_{j+1}, we seek a solution to $H_{ball} - H_{table} = 0$. Substituting $T_0 = T_j$, $V_0 = V_j$, and $y_0 = -\beta \sin(\omega T_j)$ (since at the previous impact, the ball is at the same

[2]The reader who would like to learn more about dimensional analysis should consult any basic mathematical modeling text.

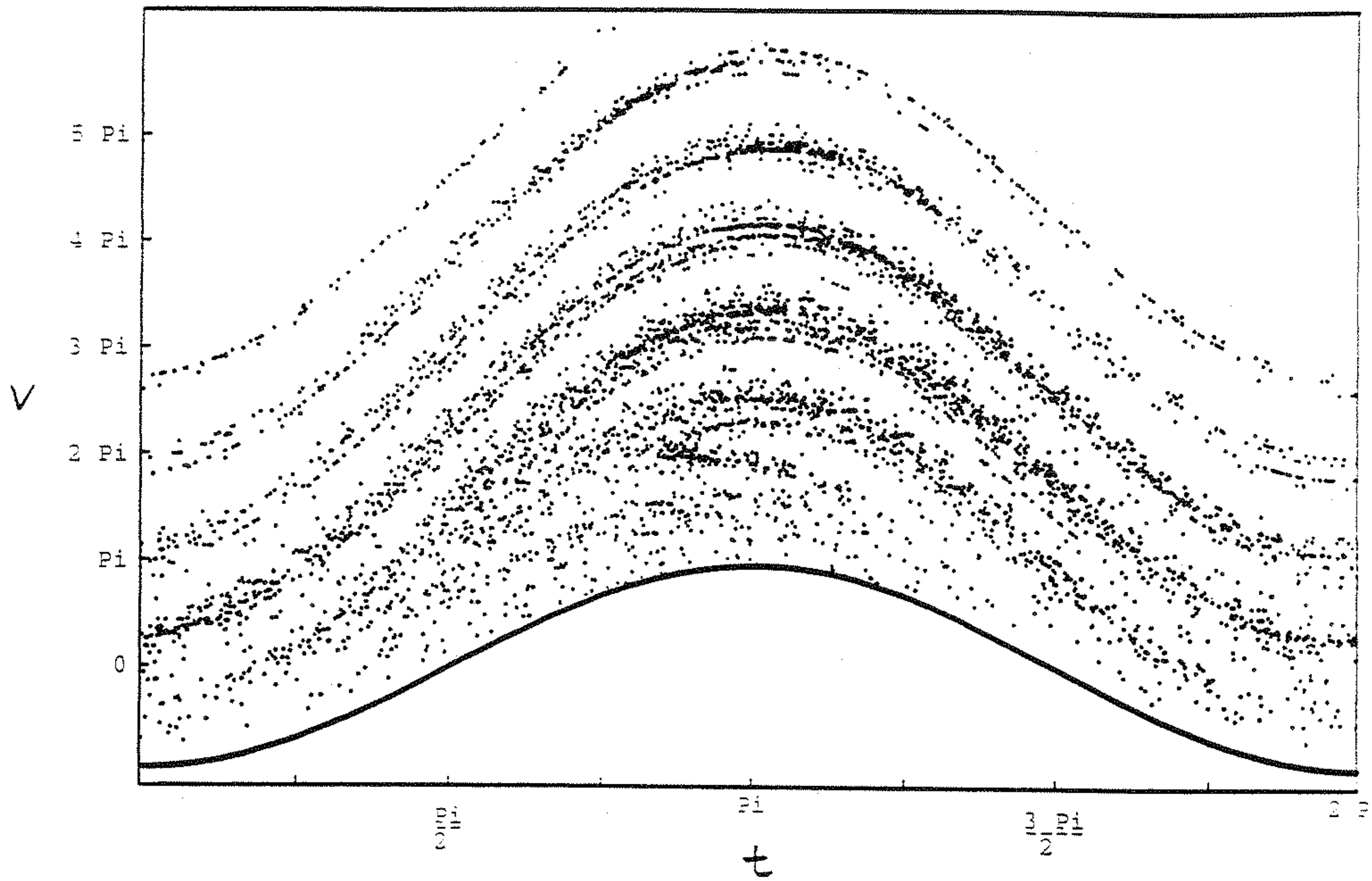


Figure 2: 7,500 iterates of the exact map for $\alpha = 0.8, a = 3$. The solid line is a plot of the velocity of the table with respect to time.

2.2 Computing the Map

To compute t_{j+1}, the time of the next impact, we are looking for the next time when the relative height of the ball with respect to the moving table is zero; i.e., when

$$h(t) = a(\sin t - \sin t_j) + v_j(t - t_j) - 1/2(t - t_j)^2 = 0. \qquad (14)$$

Since this equation cannot be solved explicitly in terms of t, we are forced to turn to numerical methods to calculate the next iterate of the map.

The problem is then this: Given v_j and t_j, find the first solution to Equation (14) such that $t > t_j$. One might think that we could do this rather easily by picking some $t_g > t_j$ in such a way that the ball is guaranteed to have hit the table by time t_g. Then we could use any root-finding technique (bisection, Newton's method, etc.) to find the solution to Equation (14) in the interval $[t_j, t_g]$. However, consider the situation in Figure 3(a). Here I have plotted the relative height function for a particular t_j and v_j. Notice that there are two solutions very close to each other; thus, if we were to choose a t_g to the right of the second solution, there would be no assurance that our root-finding method would give us the correct solution.

So how do we "home in" on the *first* root? There are (at least) two ways to solve this problem. One approach, taken by Tufillaro in writing his *Bouncing Ball* program, is to take small steps along the t-axis until the relative height function crosses the t-axis (this crossing can be found by evaluating the function at each step and watching for a sign-change in the results). Once we find this crossing, it is a simple matter to find the root near that point. In this way, we reduce the problem into one of finding an appropriate step size, one which is guaranteed to contain no more than one solution. Tufillaro chooses a new step size at each iterate of the map based on an estimate of the ball's flight time. See [5] for details.

We implemented a different method, one based on ideas from elementary calculus. We know that no more than one root lies between two successive critical points of a function. Similarly, no more than one critical point lies between two successive inflection points. Since the inflection points of a function f are the roots of its second derivative, if we can find the solutions to $f'' = 0$, we can "bracket" the critical points of f. By using a root-finding technique with these brackets, we can find the roots of f', thus determining the function's critical points and bracketing the roots of f. Then we can find any desired root of f by searching within the bracketed intervals. For a visual illustration of these ideas, consult Figure 3.

The astute reader will be asking herself, "Why must we use the second derivative at all? Why not just bracket the roots of f by finding the roots of f' straight away?" In those cases in which the roots of the first derivative can be found analytically, it is true that we need not concern ourselves with the second derivative. However, the first derivative of the relative height function is

$$h'(t) = a\cos t + v_j - (t - t_j), \tag{15}$$

which cannot be solved analytically. Fortunately for us, the second derivative,

$$h''(t) = -a\sin t - 1, \tag{16}$$

can be solved. The solutions of $h''(t) = 0$ are given by

$$t = \arcsin(-1/a) + 2n\pi, \pi - \arcsin(-1/a) + 2n\pi \tag{17}$$

with $n = 1, 2, 3, \ldots$.

With these ideas in mind, we can construct a rough outline of the algorithm:

1. Using Equation (17), find all zeroes of $h''(t)$ between t_j and some t_{max}. The zeroes thus found are the inflection points of $h(t)$.

2. Check for a sign-change in $h'(t)$ between successive inflection points. If there is a sign-change, a critical point exists between those two points, so we find it.

3. Check for a sign-change in $h(t)$ between t_j and the first critical point. If there is a sign-change, find the root between those two points. This is t_{j+1}. If there is not a sign-change, check between the first two critical points, then between the second and the third, and so on until t_{j+1} is found.

the stability of periodic orbits of the map. Recall that the Jacobian of a map in two variables x_n and y_n with the two map functions $x_{n+1} = f_1(x_n, y_n)$ and $y_{n+1} = f_2(x_n, y_n)$ is given by

$$Df = \begin{bmatrix} \partial f_1/\partial x_n & \partial f_1/\partial y_n \\ \partial f_2/\partial x_n & \partial f_2/\partial y_n \end{bmatrix}. \tag{23}$$

With the exact map, then, we wish to find

$$Df = \begin{bmatrix} \partial t_{j+1}/\partial t_j & \partial t_{j+1}/\partial v_j \\ \partial v_{j+1}/\partial t_j & \partial v_{j+1}/\partial v_j \end{bmatrix}. \tag{24}$$

Since part of the exact map is implicit, it is necessary to use implicit differentiation to calculate Df. To this end, note that if we have a function $f(x, y) = 0$, we can compute $\partial x/\partial y$ by using the following equation:

$$\frac{\partial x}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial x}. \tag{25}$$

Using this equation to compute the partials of t_{j+1} (and straightforward differentiation to get those of v_{j+1}), we get these results:

$$\frac{\partial t_{j+1}}{\partial t_j} = \frac{a\cos t_j + v_j - (t_{j+1} - t_j)}{a\cos t_{j+1} + v_j - (t_{j+1} - t_j)}, \tag{26}$$

$$\frac{\partial t_{j+1}}{\partial v_j} = \frac{-(t_{j+1} - t_j)}{a\cos t_{j+1} + v_j - (t_{j+1} - t_j)}, \tag{27}$$

$$\frac{\partial v_{j+1}}{\partial t_j} = -\alpha(1 - \partial t_{j+1}/\partial t_j) + a(1+\alpha)(\partial t_{j+1}/\partial t_j)\sin t_{j+1}, \tag{28}$$

$$\frac{\partial v_{j+1}}{\partial v_j} = -\alpha(1 - \partial t_{j+1}/\partial v_j) + a(1+\alpha)(\partial t_{j+1}/\partial v_j)\sin t_{j+1}. \tag{29}$$

Now, the determinant of Df tells us if the map contracts ($\det Df < 1$), preserves ($\det Df = 1$), or expands ($\det Df > 1$) area. The determinant of the Jacobian of the exact map is

$$\det Df = \alpha\left(\frac{\partial t_{j+1}}{\partial v_j} - \frac{\partial t_{j+1}}{\partial t_j}\right). \tag{30}$$

For the fixed points of the map, $\partial t_{j+1}/\partial t_j = 1$ and $\partial t_{j+1}/\partial v_j = 1+\alpha$, so $\det Df = \alpha^2$ for these points.[3] This is an interesting fact, for in the Holmes map, $\det Df$ was always α. This result indicates that in some cases, the exact map contracts

[3]In connection to these values for the determinants, there is an interesting note about the coefficient of restitution. While the heights of successive bounces are reduced by a factor of α, the *energy* of the ball is reduced by a factor of α^2. Since the determinant of the Jacobian can indicate how a system dissipates energy, this may be a corellation between the physical system and the mathematical analysis, or it may simply be a coincidence.

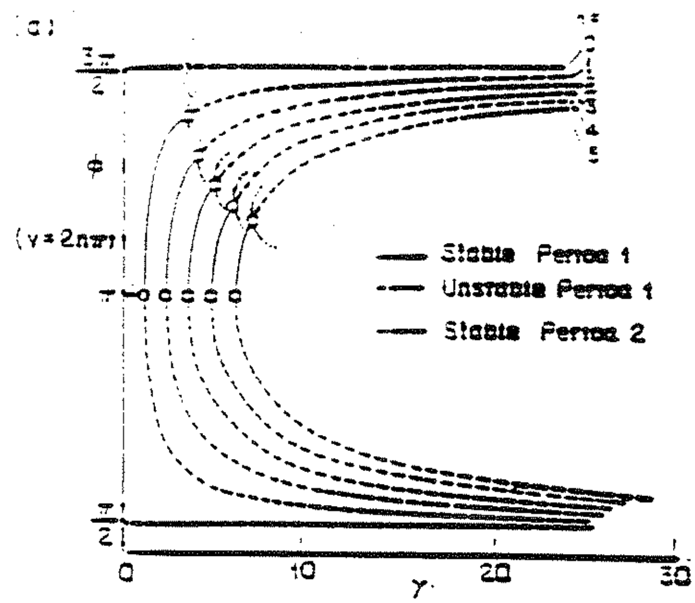


Figure 4: A bifurcation diagram for the bouncing ball system. Here we see the saddle-node bifurcations that give rise to the periodic orbits and the period-doubling bifurcations that lead to the demise of the stable orbits. From [3].

This inequality is neccesary for a periodic orbit to exist but not sufficient, as we will see in §3.2.

Physically, this can be interpreted as the a value at which the table gives the ball just enough "kick" to compensate for the energy dissipated at the impact. At the critical value $a = n\pi(1-\alpha)/(1+\alpha)$, there exists one periodic orbit at $t_j = \pi$. This is just the value of t_j at which we would expect an orbit to first appear, for π is the point of maximum velocity for the table, and, subsequently, maximum "kick." As a increases, this single orbit splits off immediately into a stable and an unstable orbit which approach $3\pi/2$ and $\pi/2$, respectively, as a goes to infinity. This behavior is illustrated in Figure 4.

3.2 Glancing Collisions and the Death of the Periodic Orbits

Figure 4 doesn't tell the whole story, however. While Holmes's diagram indicates the disappearance of the *stable* orbits, it leads one to believe that the *unstable* orbits persist indefinitely. This is not the case, for these orbits are destroyed by the glancing collisions cited in the discussion of the algorithm for computing the map.

As mentioned before, as a increases, t_j for the unstable orbit approaches $\pi/2$. More importantly, though, at the same time, the height of the table's peak is increasing. As this height increases, the ball's trajectory passes over the table's by an ever-decreasing amount until at some critical value the ball just barely tuting into Equations (38) and (39), we have

$$0 = a_1^*(\sin t^* - 1) + n\pi(t^* - \pi/2) - 1/2(t^* - \pi/2)^2, \tag{40}$$

$$0 = a_1^* \cos t^* + n\pi - (t^* - \pi/2). \tag{41}$$

When $n = 1$, we know by symmetry that $t^* = 3\pi/2$, so Equation (40) reduces further to

$$-2a_1^* - \pi^2/2 = 0. \tag{42}$$

Thus, for $n = 1$, $a_1^* = \pi^2/4$. For other n, the analysis is more difficult, and we must resort to numerical solutions.

What happens as α gets smaller? One might guess that there exists an α such that the first periodic orbit, born at the saddle-node bifurcation at $(t_j, v_j) = (\pi, n\pi)$, is destroyed immediately by a glancing collision. This is indeed the case, and numerical solutions indicate that this happens at $\alpha = .020457\ldots$ for $n = 1$. At this point, the glancing bifurcation changes nature: for α less than this value, the unstable periodic orbit doesn't exist at all, and the stable periodic orbit is not born at the saddle-node bifurcation. Instead, it is created when the table has enough energy to allow the ball to clear the peak of the table's oscillation, i.e., when the glancing collision is avoided. Thus, for $\alpha < .020457\ldots$, a_α^* is no longer that a value at which the unstable periodic orbit is destroyed; rather, it is the a value at which the stable periodic orbit reappears.

With this in mind, we want to determine the value of a_0^*. For this α value, the ball gives all of its energy to the table and does not bounce at all. Thus, we are looking for a range of a values at which the ball will be repeatedly *thrown* from the table in a periodic manner. If a ball is at rest on the table, it departs when the table's downward acceleration is faster than the ball's rate of freefall under the influence of gravity, i.e., when the acceleration of the table is less than or equal to -1.[4] Therefore, for $\alpha = 0$, periodic orbits exist when

$$a \sin t_j \leq -1, \tag{43}$$

where t_j is given by Equation (34).

From Equation (34), we know that

$$\cos t_j = \frac{n\pi(\alpha - 1)}{a(1 + \alpha)} \tag{44}$$

for the periodic orbits. Remembering from high school trigonometry that $\sin t = \pm\sqrt{1 - \cos^2 t}$, we can substitute, simplify, and rewrite Equation (43) as

$$a \left(\pm\sqrt{1 - \frac{a^2 - n^2\pi^2}{a^2}} \right) \leq -1. \tag{45}$$

Since we are concerned with the stable periodic orbits, we can replace the $\pm$ in Equation (45) with a $-$. Doing so and simplifying, we have

$$a^2 - n^2\pi^2 \geq 1 \tag{46}$$

[4] Remember that in our dimensionless equations, $g = 1$.

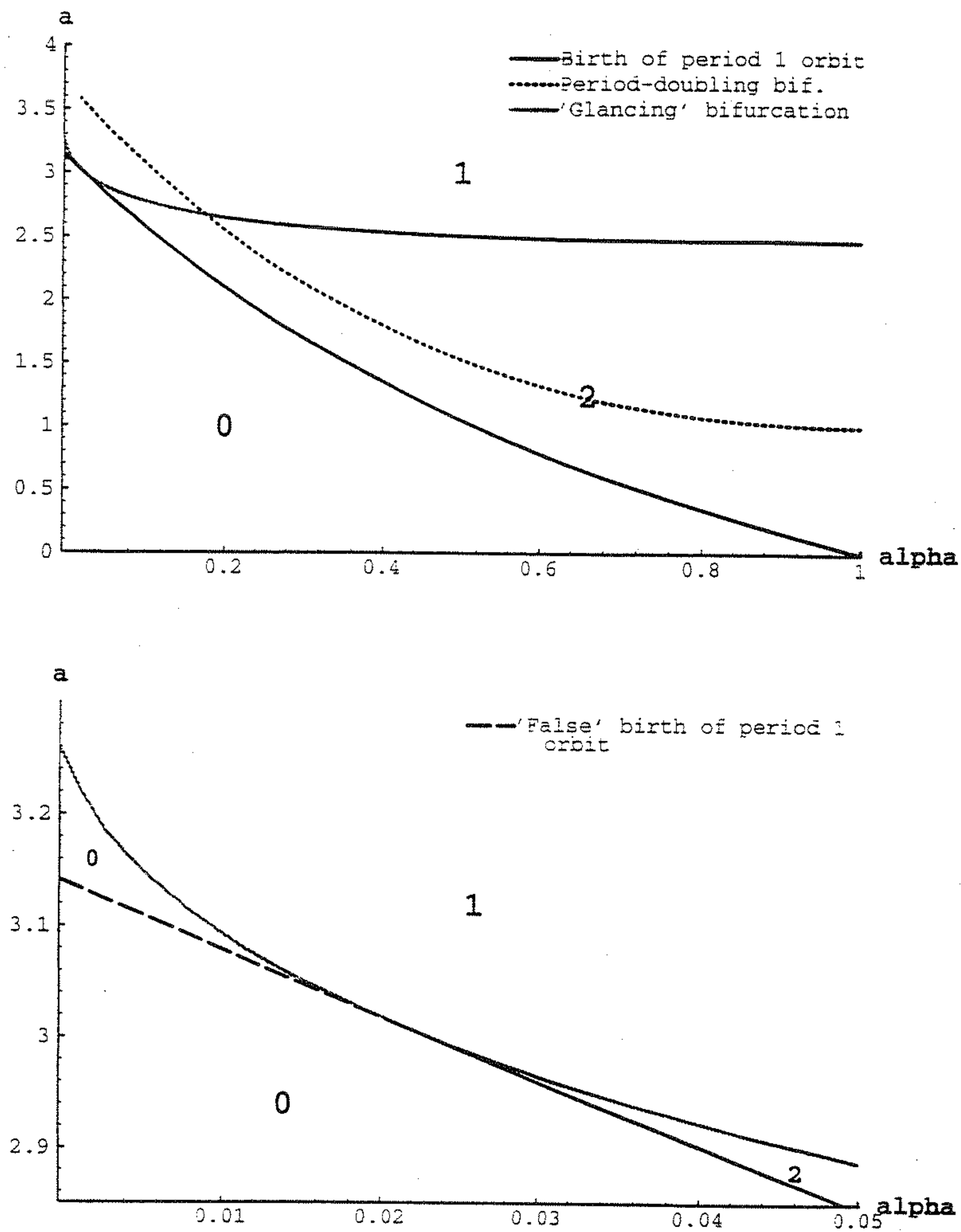
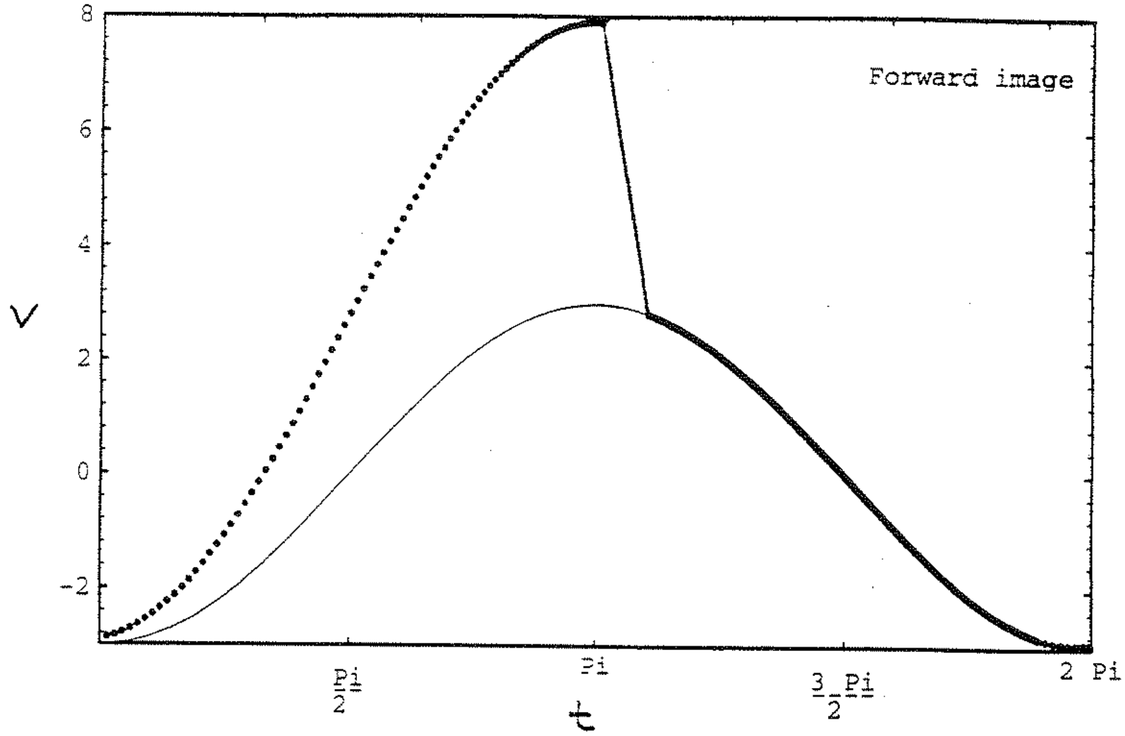


Figure 7: A bifurcation diagram for the exact map, indicating different forms of births and deaths of the periodic orbits. The bold-faced numbers indicate the number of periodic orbits that exist in that region. Figure (b) is a close-up of the region near the point where the glancing bifurcation changes nature (see page 13).



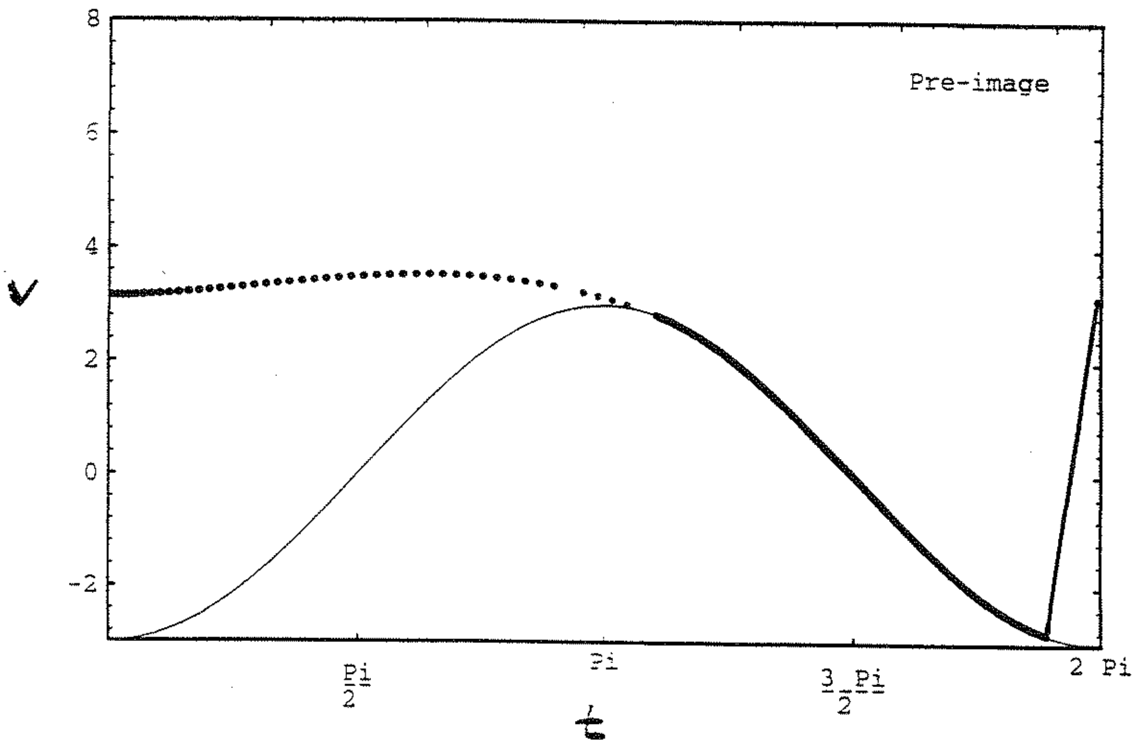


Figure 8: Plots of the forward and pre-image of the glancing collisions for $a = 3$. The solid curve in each figure is a plot of the velocity of the table's oscillation, and the bold line indicates the interval on which glancing collisions are possible. The dotted lines are the images, with lines connecting the endpoints of the bold line to the points they map to (or from) on the dotted line.

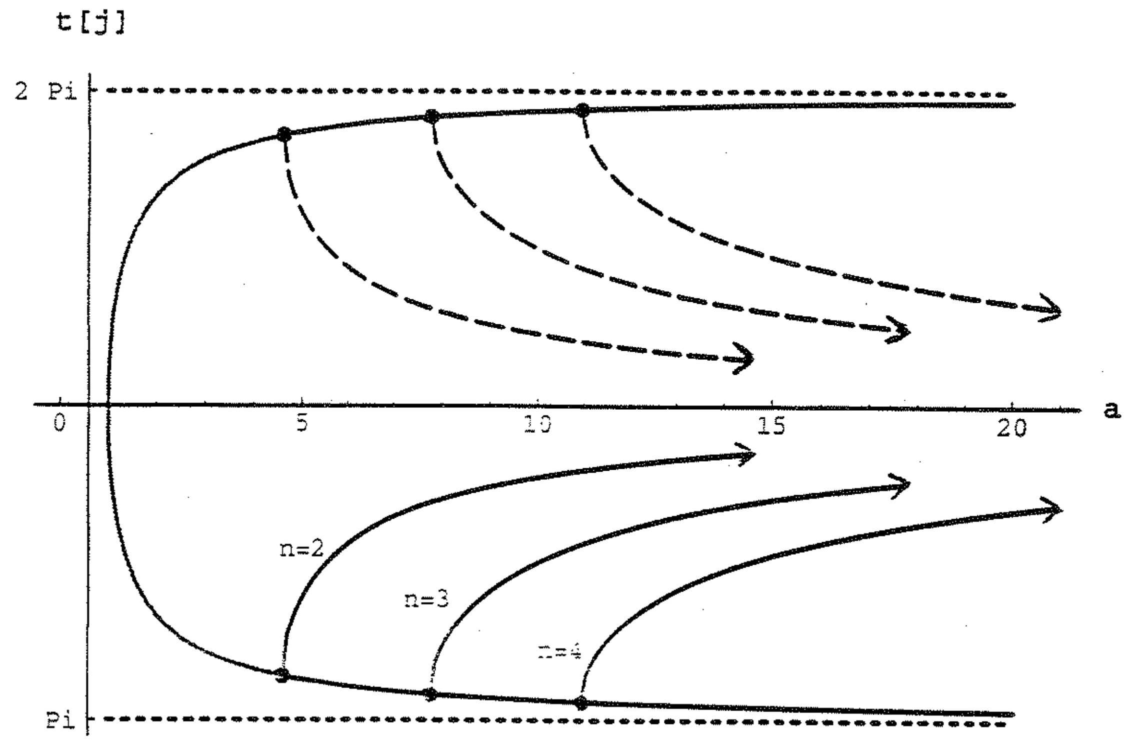


Figure 10: A plot of t_j vs. a indicating the points at which the discontinuities in the forward and pre-images appear. The gray line forms the boundary of the region in which glancing collisions are possible; a throw-and-catch occurs at each solid dot; trajectories that leave from a glancing collsion on the solid line map to another glancing collision (while trajectories that map to glancing collisions on the dotted lines are glancing collisions themselves). The values of n indicate how many oscillations of the table the trajectories pass over.

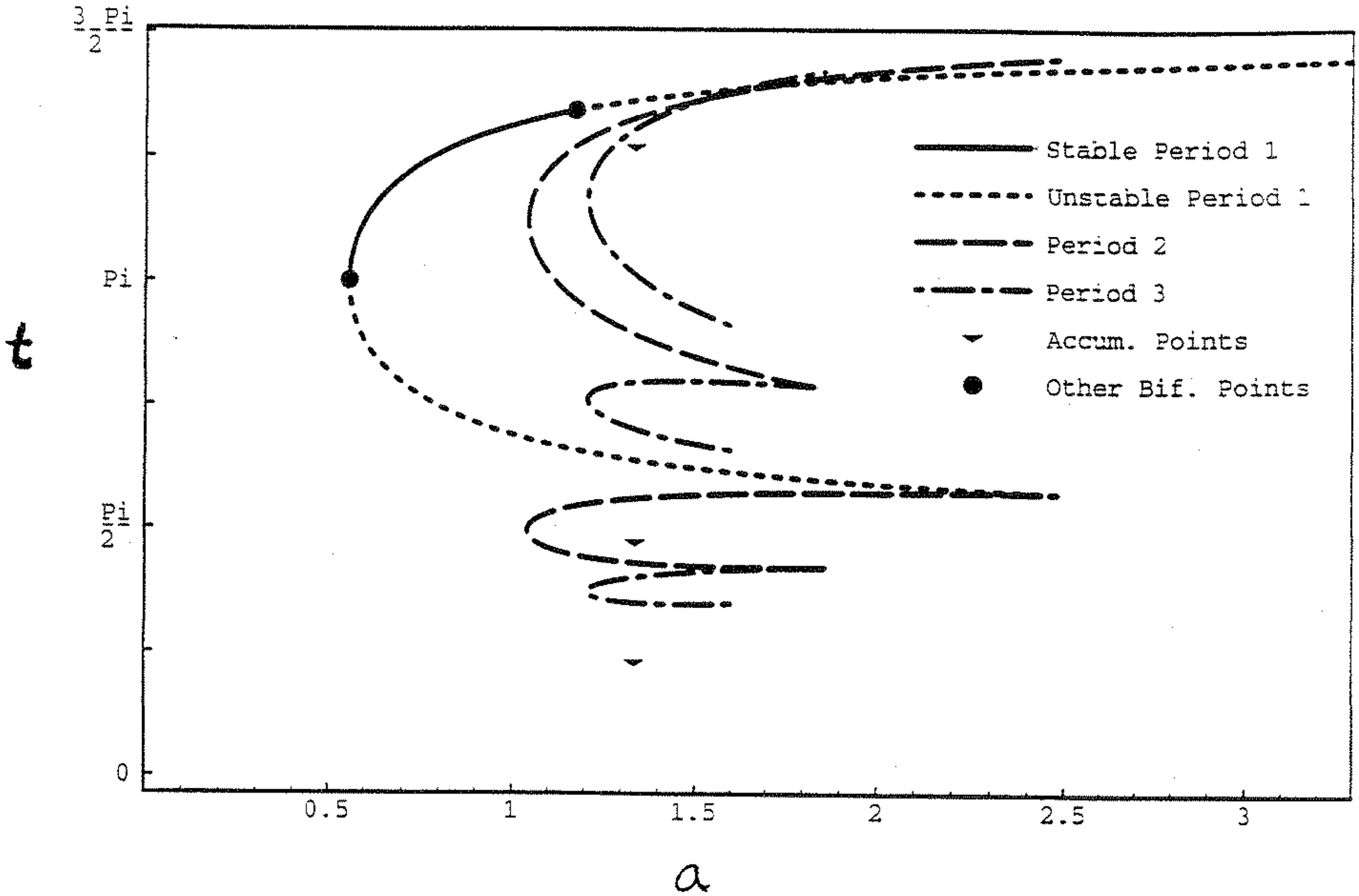


Figure 11: A bifurcation diagram showing the initial birth of the unstable period-1 orbit and then the collsions of successively higher-period orbits, eventually accumulating to an infinite-periodic orbit ($\alpha \approx .7$). Only three of the infinite number of these accumulation points are shown: the throw (the top point) and the next two impacts. The "other bifurcation points" are the saddle-node bifurcation and the period-doubling bifurcation of the period-1 orbit.

References

[1] Richard J. Gaylord, Samuel N. Kamin, and Paul P. Wellin. *Introduction to Programming with Mathematica.* Springer-Verlag: New York, 1993.

[2] John Guckenheimer and Phillip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag: New York, 1983.

[3] Phillip J. Holmes. "The Dynamics of Repeated Impacts With a Sinusoidally Vibrating Table." *J. Sound Vib.* , 84, 173–89.

[4] William H. Press, Brian T. Flannery, Saul A. Teukolsky, and William T. Vetterling. *Numerical Recipes in C: The Art of Scientific Computing.* Cambridge University Press: Cambridge, 1988.

[5] Nicholas B. Tufillaro, Tyler Abbott, and Jeremiah Reilly. *An Experimental Approach to Nonlinear Dynamics and Chaos.* Addison-Wesley Publishing Company: Redwood City, CA, 1992.