Orlik-Solomon Algebras and Tutte Polynomials

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Abstract

An **arrangement** $\mathcal{A}$ is a finite set of hyperplanes in $\mathbb{C}^\ell$. Associated with $\mathcal{A}$ is a graded algebra A($\mathcal{A}$) called the **Orlik-Solomon algebra**, whose definition is motivated by topological considerations. Let G be a connected simple graph. There is a natural way to construct an arrangement $\mathcal{A}_G$ from the graph G, and the algebra associated with $\mathcal{A}_G$ depends only on G. More generally, the Orlik-Solomon algebra depends only on the underlying matroid of the arrangement. The additive structure of A($\mathcal{A}_G$) is uniquely determined by the chromatic polynomial of G.

There are very few examples of non-isomorphic matroids whose Orlik-Solomon algebras are isomorphic. In each known case, the matroids have the same Tutte polynomial. The **Tutte polynomial** is a two-variable generalization of the chromatic polynomial which carries much information. It is natural to conjecture that the Tutte polynomial is an invariant of the algebra.

We construct, for any graphic arrangement $\mathcal{A}_G$, an infinite family of pairs of graphic arrangements containing $\mathcal{A}_G$ each of which have isomorphic Orlik-Solomon algebras, but different Tutte polynomials.

Section 1

Definition 1.1

Given a graph G, a **loop** is an edge that takes a vertex to itself:

Figure 1

An **isthmus** is an edge such that, if taken away, the graph becomes disconnected or more disconnected:

Figure 2

Definition 1.2

Given a graph G, and a set E consisting of the edges of G (called the edge set of G) a **matroid** on G is E together with a set I, which is all subsets of E that correspond to subgraphs of G that are forests (contain no cyclic graphs). It is denoted $M(E, I)$ or M_G. (see Oxley) We will use G and M_G interchangeably.

Example 1.

E={1,2,3,4,5,6}

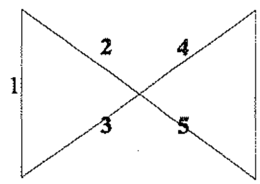


Figure 3

I=

all	**124**	**136**	**235**	**345**	**1256**	**2346**
singleton	**125**	**145**	**236**	**346**	**1345**	**2356**
sets,	**126**	**146**	**245**	**356**	**1346**	
all sets of	**134**	**156**	**246**	**1245**	**1356**	
size 2,	**135**	**234**	**256**	**1246**	**2345**	

More simply put, if a subset I_1 of E does not contain the subset 123 or 456, then I_1 is in I. We are more concerned with the cyclic subgraphs of G--123 and 456. In matroid terminology, the subsets of E corresponding to cyclic subgraphs of G are called **circuits**, and the set of circuits is called C. Because elements of I cannot contain a cyclic subset, I can be determined by C, and so the matroid can be determined by E and C.

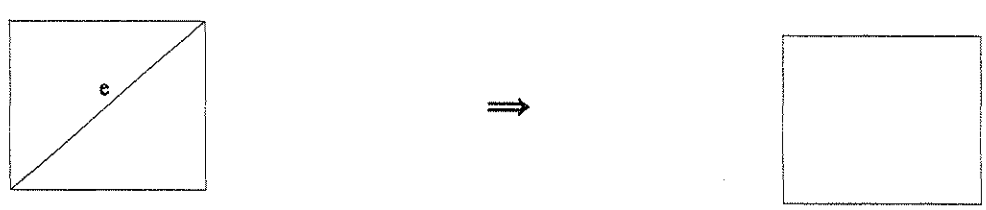
Definition 1.3

Given a matroid M, the **Tutte polynomial** of M, denoted $T(M; x, y)$ or T_M, is a polynomial in the variables x and y, such that

*T*1) $T(\text{loop}) = y$; $T(\text{isthmus}) = x$
*T*2) $T(M; x, y) = T(M\backslash e; x, y)T(e)$ if e is a loop or isthmus
*T*3) $T(M; x, y) = T(M\backslash e; x, y) + T(M/e; x, y)$ otherwise.

It is a matroid isomorphism invariant, meaning $M_1 \cong M_2$ implies $T_{M1}=T_{M2}$. (see Brylawski and Oxley)

Property *T*3) is called deletion-contraction. $M\backslash e$ refers to the **deletion** of e from E -- the graph G without that edge.



G $G\backslash e$

Figure 4

M/*e* refers to the **contraction** of *e* in *G* -- *e* is deleted, and the endpoint vertices of *e* are identified with each other. This, however, is easiest to explain pictorially.

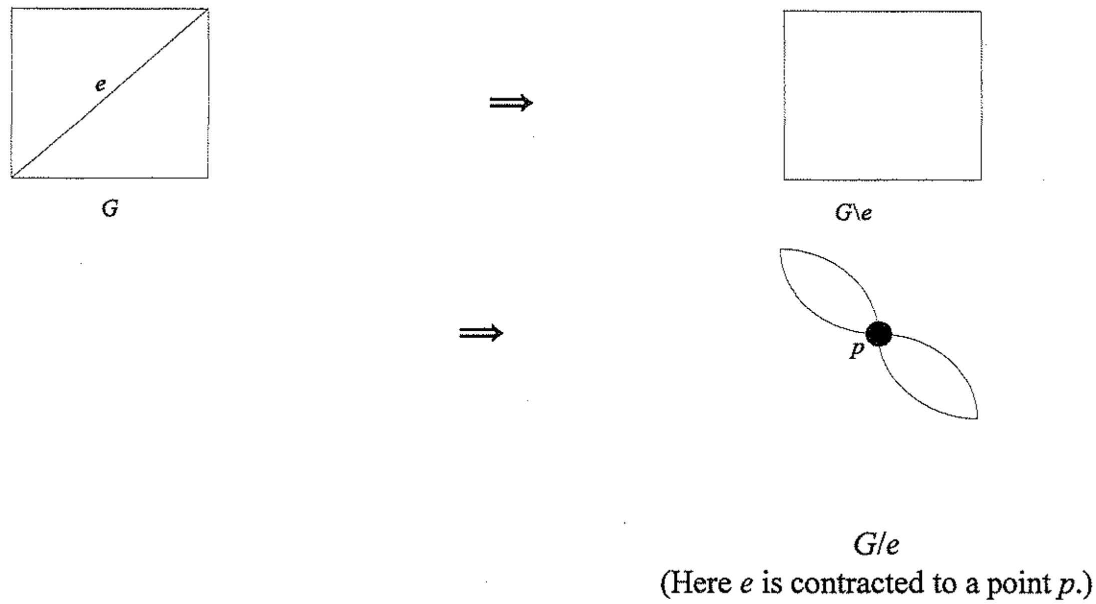
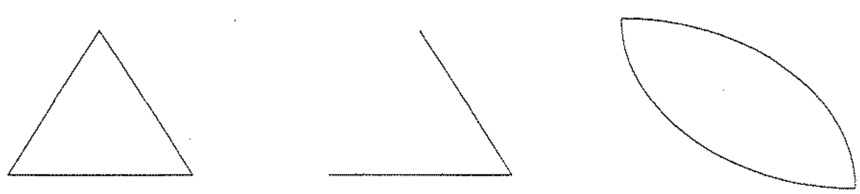


Figure 5

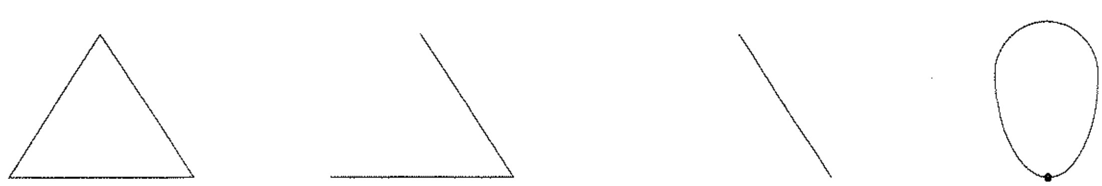
Example 2. To find T_{C3}, deletion-contraction needs to be applied, since C_3 contains no loops or isthmuses.



$$T_{C3} = T_{C3\backslash e} + T_{C3/e}$$

Figure 6

Since $C_3\backslash e$ consists of two isthmuses, $T_{C3\backslash e}=x^2$ by applying *T*1 and *T*2. $T_{C3/e}$ contains no loops or isthmuses, so deletion-contraction is again applied. We let *G* be C_3/e.



$$T_{C3} = x^2 + T_{G\backslash e} + T_{G/e}$$

Figure 7

$G \backslash e$ is an isthmus, so $T_{G\backslash e}=x$. G/e is a loop, so $T_{G/e}=y$. Therefore, $T_{C3}=x^2+x+y$.

Example 3. Let G be

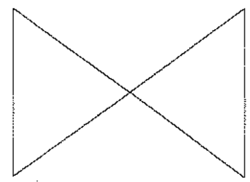
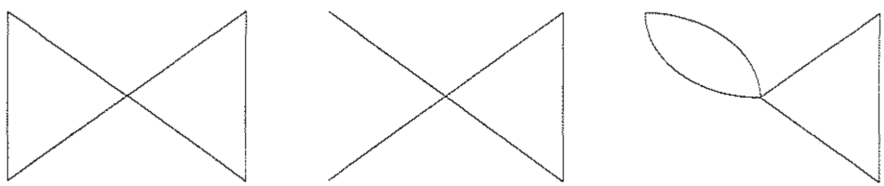


Figure 8

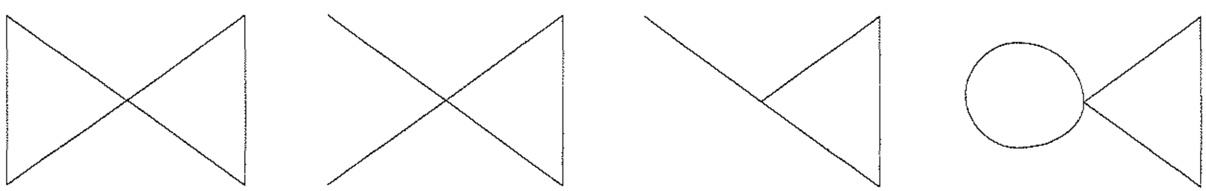
T_G contains no loops or isthmuses, so apply deletion-contraction.



$T_G \quad = \quad T_{G\backslash e} \quad + \quad T_{G/e}$

Figure 9

Notice that $G \backslash e$ consists of C_3 and two isthmuses. By applying *T2* twice, $T_{G\backslash e}=x^2(T_{C3})$. G/e contains no loops or isthmuses, so apply deletion-contraction again. We let A be G/e.



$T_A \quad = \quad x^2(T_{C3}) \quad + \quad T_{A\backslash e} \quad + \quad T_{A/e}$

Figure 10

$A \backslash e$ consists of an isthmus and C_3. By applying *T2*, $T_{A\backslash e}=x(T_{C3})$. A/e consists of an isthmus and C_3. By applying *T2*, $T_{A/e}=y(T_{C3})$. $T_G=x^2(T_{C3})+x(T_{C3})+y(T_{C3})=(x^2+x+y)(T_{C3})=(T_{C3})(T_{C3})$.

Definition 1.4

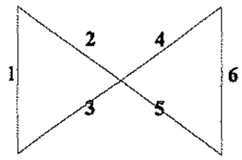
Given a matroid M with edge set $E=\{e_1, e_2, ..., e_n\}$, the **exterior algebra**, denoted $\wedge(M)$ or $E(M)$, is the algebra with basis E such that $e_i \wedge e_j = -e_j \wedge e_i$. (^ represents the product of two elements in $E(M)$.) This leads to $e_i^2=e_i \wedge e_i=0$. $e_i \wedge e_j$ will be denoted e_{ij} for simplicity. In general, where we have $x, y \in E(M)$, and x and y are of deg(x) and deg(y), $x \wedge y=(-1)^{(\deg(x))(\deg(y))}y \wedge x$.

An algebra is a vector space that is also a ring. It is easiest to picture an algebra as the structure associated with addition, subtraction, and multiplication of polynomials with real coefficients.

The **Orlik-Solomon algebra,** A(M), is the exterior algebra with basis E (the edge set of M) and with the restriction that wherever e_{i1}, e_{i2}, ..., e_{ip} is a circuit of M, $\partial(e_{i1} \wedge e_{i2} \wedge \cdots \wedge e_{ip}) = \partial(e_{i1i2\cdots ip}) = 0$. ∂ refers to the boundary of that circuit, and

$$\partial(e_{i1i2\cdots ip}) = \sum_{j=1}^{p} (-1)^{j-1} e_{i1\cdots \hat{ij} \cdots ip}$$

Example 4.



$$\partial(e_{123}) = e_{23} - e_{13} + e_{12}$$

Figure 11

Definition 1.5

Given a graph G, the **chromatic polynomial** of G, denoted $\Psi(G; t)$ or Ψ_G, is the number of proper colorings of G in t colors, or t-colorings of G.

A coloring of G is proper if the colors of the endpoints of an edge are colored differently:

This means a loop has <u>no</u> proper colorings.

Figure 12

If G contains a loop, $\Psi_G=0$. If e is a multiple edge, $\Psi_G=\Psi_{G\backslash e}$.

Example 5. Let G be

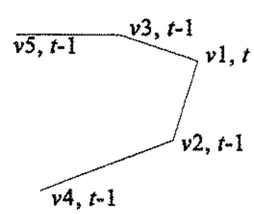


Figure 13

Choose a vertex v_1. There are t possible colors for v_1. Each of the two vertices connected to v_1, v_2 and v_3, have t-1 possible colors. The remaining vertex connected to v_2, v_4, is not connected to any vertex besides v_2, so it has t-1 possible colors. We follow similar reasoning to conclude that the final vertex, v_5, has t-1 possible colors. Determining the number of possible t-colorings becomes a simple counting problem from this point; $\Psi_G=t(t-1)^4$. In general, where G is a path with n vertices, $\Psi_G=t(t-1)^{n-1}$.

Example 6. Let G be

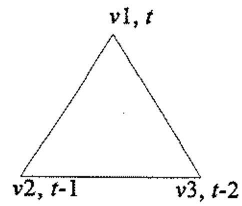


Figure 14

Choose a vertex v_1. The number of possible colors for that vertex is t. Choose another vertex v_2. Because v_2 is connected to v_1, it cannot be the same color as v_1, so there are t-1 possible colors for v_2. The remaining vertex, v_3, is connected to both v_1 and v_2, so there are t-2 possible colors for that v_3. Thus, $\Psi_G=t(t-1)(t-2)$.

If a graph G is small and uncomplicated, such as C_3, Ψ_G is fairly easy to determine. One would think that Ψ_{C4}, shown here,

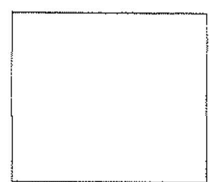


Figure 15

would be equally easy to determine. It seems that $\Psi_{C4}=t(t-1)(t-2)(t-3)$. However, because opposite corners in this graph can be the same color, this is not the case. To determine the chromatic polynomial of these more complicated graphs, there is a deletion-contraction formula for chromatic polynomials.

This formula is given by $\Psi_G=\Psi_{G\backslash e}-\Psi_{G/e}$, for a graph G. The 'easy' proof of this formula is "different = all - same." In the deletion of e from G, the two vertices, u and v that were connected by e, and therefore must be different colors, are no longer connected and so may be the same or different colors. In the contraction of e from G, u and v are the same vertex, and are therefore the same color. Hence, "different = all - same."

Example 7. Let G be

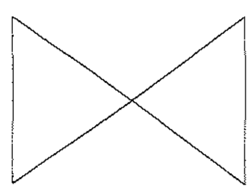
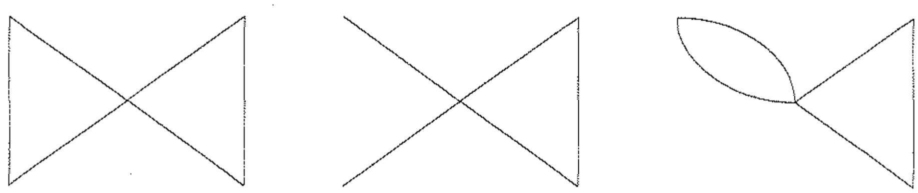


Figure 16

We apply deletion-contraction to determine Ψ_G.



$$\Psi_G \quad = \quad \Psi_{G\backslash e} \quad - \quad \Psi_{G/e}$$

Figure 17

In finding the chromatic polynomial of $G\backslash e$ and G/e, we revert to the previous method of determination. Choosing the center vertex v in $G\backslash e$ to have t possible colors, the vertices connected to it by isthmuses each can have t-1 possible colors, since neither are connected to any other vertex. The colorings of the other two vertices are t-1 and t-2, as above in Example 6. Thus, $\Psi_{G\backslash e}=t(t-1)^3(t-2)$. Following the same logic for G/e and the knowledge that multiple edges do not affect the chromatic polynomial, $\Psi_{G/e}=t(t-1)^2(t-2)$. Therefore, $\Psi_G=t(t-1)^3(t-2)-t(t-1)^2(t-2)$, or, upon simplification, $\Psi_G=t(t-1)^2(t-2)^2$. Notice, that $\Psi_G=(\Psi_{C3})(\Psi_{C3})/t$.

Definition 1.6

Given a matroid M, the **characteristic polynomial** of M, $\mathbf{X}(M;\,t)$ or $\mathbf{X}_M$, is the Tutte polynomial of M with x evaluated at 1-t and y evaluated at 0, or $\mathbf{X}_M=\mathrm{T}(M;\,1\text{-}t;\,0)$. Like the Tutte polynomial, the characteristic polynomial is a matroid isomorphism invariant. The characteristic polynomial is, within a factor of t, equal to the chromatic polynomial. More specifically, where G is the underlying graph of a matroid and k_G is the number of connected components of G, $\Psi_G=t^{kG}\mathbf{X}_G$. Since we will be dealing solely with connected graphs, $k_G=1$ and therefore $\Psi_G=t\mathbf{X}_G$ for the purposes of this paper. (see Zaslavsky)

We know that non-isomorphic matroids may have isomorphic Orlik-Solomon algebras ($\mathrm{A}(M_1)\cong\mathrm{A}(M_2)$ does not imply $M_1\cong M_2$). The matroid examples presented here show this as well. The question addressed here is:

Do matroids with isomorphic Orlik-Solomon algebras have equal Tutte polynomials? (Does $\mathrm{A}(M_1)\cong\mathrm{A}(M_2)$ imply $\mathrm{T}_{M1}=\mathrm{T}_{M2}$?)

In order to answer this question, we need to understand the isomorphism $\mathrm{A}(M_1)\cong\mathrm{A}(M_2)$. We consider $\mathrm{A}(M_1)\cong\mathrm{A}(M_2)$ when $\Psi_{M1}=\Psi_{M2}$ and there exists a Φ: $\mathrm{E}(M_1)\rightarrow\mathrm{E}(M_2)$ such that Φ is bijective and takes the circuits of M_1 to the circuits of M_2. If $\mathrm{A}(M_1)\cong\mathrm{A}(M_2)$, we then find T_{M1} and T_{M2} and determine if $\mathrm{T}_{M1}=\mathrm{T}_{M2}$.

Section 2

Let $C_3\bullet C_3$ and $C_3|C_3\bullet C_2$ (C_2 is used here to represent the isthmus on the graph. It is not standard notation.) be the following graphs. (The second graph is called a parallel construction, but no standard notation has been found for it in the literature.)

$C_3 \bullet C_3$ $\quad$ $C_3|C_3 \bullet C_2$

Figure 18

As has been shown above, $\Psi_{C3 \bullet C3}=t(t-1)^2(t-2)^2$. Notice that $\Psi_{C3 \bullet C3}=(\Psi_{C3})^2/t$.

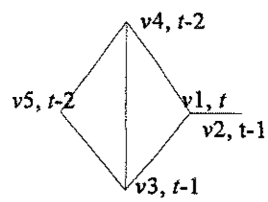


Figure 19

$\Psi_{C3|C3 \bullet C2}$ can easily be found. Choose a vertex v_1. There are t possible colors for v_1. There are t-1 possible colors for the vertex v_2, which is connected to v_1 by the isthmus. Choose another vertex v_3 that is also connected to v_1. There are t-1 possible colors for v_3. There are t-2 possible colors for v_4, a vertex connected to both v_1 and v_3. The remaining vertex, v_5, is connected to both v_3 and v_4, so there are t-2 possible colors for v_5. Therefore, $\Psi_{C3|C3 \bullet C2} = t(t-1)^2(t-2)^2=\Psi_{C3 \bullet C3}$.

These examples have the same chromatic polynomials. We will now show that this is the case for general pairs of graphs, $C_n \bullet C_m$ and $C_n|C_m \bullet C_2$.

Theorem 2.1. *Let A and B be graphs.* $\Psi_{A \bullet B}=(\Psi_A)(\Psi_B)/t$.

Proof:
Let G_1 be the graph $A \bullet B$. Let G_2 be the graph formed by the disjoint graphs A and B. In the case where A is C_3 and B is C_3, G_1 and G_2 are respectively

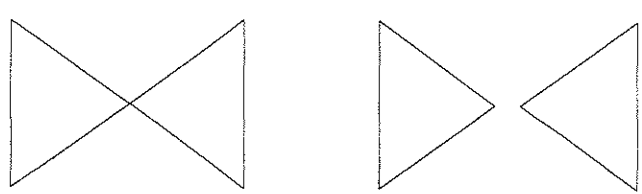


Figure 20

It is obvious that the matroids formed by G_1 and G_2 have the same set of circuits, and so are isomorphic. Since the characteristic polynomial is an isomorphism invariant of the matroid, $X_{G1}=X_{G2}$. $k_{G1}=1$, so $\Psi_{G1}=tX_{G1}$. Similarly, $k_{G2}=2$, so $\Psi_{G2}=t^2X_{G2}$. $\Psi_{G1}/t=\Psi_{G2}/t^2$ and thus

$\Psi_{G1}=\Psi_{G2}/t$. Ψ_{G2} is determined by Ψ_A and Ψ_B. These are independent of each other since A and B are disjoint. Then $\Psi_{G2}=(\Psi_A)(\Psi_B)$, and therefore $\Psi_{A\bullet B}=(\Psi_A)(\Psi_B)/t$.

In our case, if A is C_n and B is C_m, then $\Psi_{Cn\bullet Cm}=(\Psi_{Cn})(\Psi_{Cm})/t$.

It is a more involved task to show that $\Psi_{Cn|Cm\bullet C2}=(\Psi_{Cn})(\Psi_{Cm})/t$. We first prove that

$$\Psi_{Cn}=(-1)^n(t-1)[1-(1-t)^{n-1}]\ \forall n\geq 3.$$

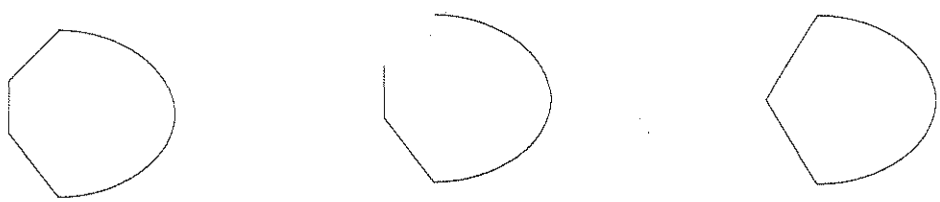
Lemma 2.2. $\Psi_{cn}=(-1)^n(t-1)[1-(1-t)^{n-1}]\ \forall\ n\geq 3.$

Proof: $\Psi_{C3}=t(t-1)(t-2)$, as shown above in Example 6.

$$\begin{aligned}(-1)^3(t-1)[1-(1-t)^{3-1}]&=(-1)(t-1)[1-(1-t)^2]\\&=(-1)(t-1)[1-(1-2t+t^2)]\\&=(-1)(t-1)(2t-t^2)\\&=t(t-1)(t-2)=\Psi_{C3}\end{aligned}$$

$$\Psi_{Cn}=(-1)^n(t-1)[1-(1-t)^{n-1}]\rightarrow\Psi_{Cn+1}=(-1)^{n+1}(t-1)[1-(1-t)^n]$$

Assume $\Psi_{cn}=(-1)^n(t-1)[1-(1-t)^{n-1}]$. By deletion-contraction,



$$\Psi_{C_{n+1}} = \Psi_{C_{n+1}\backslash e} - \Psi_{C_{n+1}/e}$$

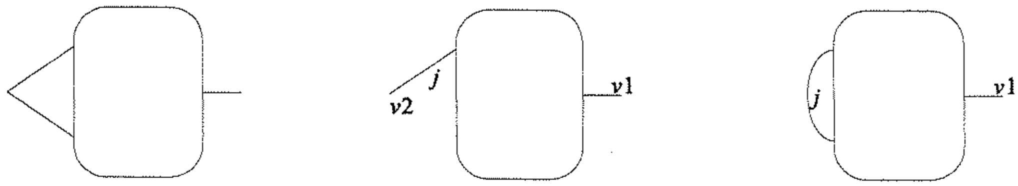
Figure 21

$C_{n+1}\backslash e$ is a path with n edges and $n+1$ vertices since C_{n+1} has $n+1$ edges and $n+1$ vertices. $C_{n+1}\backslash e$ is a path with n edges, so, as shown above in Example 5, $\Psi_{cn+1\backslash e}=t(t-1)^n$. For similar reasons, C_{n+1}/e is C_n, so $\Psi_{C_{n+1}/e}=\Psi_{C_n}$.

$$\begin{aligned}\text{Then, }\Psi_{Cn+1}&=t(t-1)^n-\Psi_{Cn}\\&=t(t-1)^n-(-1)^n(t-1)[1-(1-t)^{n-1}],\ \textit{by our assumption.}\\&=(t-1)\left(t(t-1)^{n-1}+(-1)^{n+1}[1-(1-t)^{n-1}]\right)\\&=(t-1)\left((-1)^{n-1}t(1-t)^{n-1}+(-1)^{n+1}[1-(1-t)^{n-1}]\right)\\&=(-1)^{n+1}(t-1)\left((-1)^{-2}t(1-t)^{n-1}+1-(1-t)^{n-1}\right)\\&=(-1)^{n+1}(t-1)\left(1+t(1-t)^{n-1}-(1-t)^{n-1}\right)\\&=(-1)^{n+1}(t-1)\left(1+(t-1)(1-t)^{n-1}\right)\\&=(-1)^{n+1}(t-1)[1-(1-t)^n]\end{aligned}$$

By induction, $\Psi_{Cn}=(-1)^n(t-1)[1-(1-t)^{n-1}]$.

Theorem 2.3. *Let G be a graph.* $\Psi_{Cn|G\bullet C2}=(\Psi_{Cn})(\Psi_G)/t$.



$\Psi_{C3|G\bullet C2} \quad = \quad \Psi_{C3|G\bullet C2\backslash e} \quad - \quad \Psi_{C3|G\bullet C2/e}$

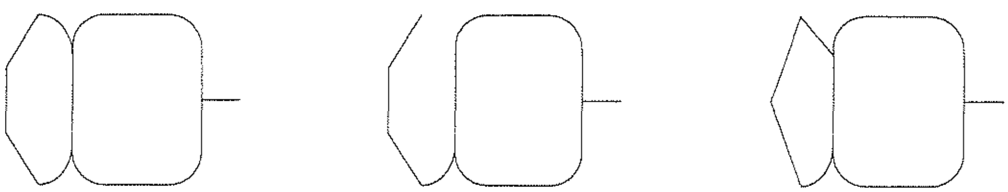
Figure 22

Proof: In $C_3|G\bullet C_2\backslash e$, vertices v_1 and v_2 are each connected to one vertex in G, so there are t-1 possible colors for v_1 and t-1 possible colors for v_2. So $\Psi_{C3|G\bullet C2}=(t-1)^2\Psi_G$. In $C_3|G\bullet C_2/e$, there are t-1 colors for v_1, for similar reasons as in $C_3|G\bullet C_2\backslash e$. Also, because edge j in $C_3|G\bullet C_2/e$ is contracted to a multiple edge, it does not affect $\Psi_{C3|G\bullet C2/e}$. So, $\Psi_{C3|G\bullet C2/e}=(t-1)\Psi_G$.

$$\begin{aligned}\Psi_{C3|G\bullet C2}&=\Psi_{C3|G\bullet C2\backslash e}-\Psi_{C3|G\bullet C2/e}\\ &=(t-1)^2\Psi_G-(t-1)\Psi_G\\ &=(t-1)(t-2)\Psi_G\\ &=t(t-1)(t-2)\Psi_G/t\\ &=(\Psi_{C3})(\Psi_G)/t\end{aligned}$$

$$\Psi_{Cn|G\bullet C2}=(\Psi_{Cn})(\Psi_G)/t \rightarrow \Psi_{Cn+1|G\bullet C2}=(\Psi_{Cn+1})(\Psi_G)/t$$

Assume $\Psi_{Cn|G\bullet C2}=(\Psi_{Cn})(\Psi_G)/t$. By deletion-contraction,



$\Psi_{Cn+1|G\bullet C2} \quad = \quad \Psi_{Cn+1|G\bullet C2\backslash e} \quad - \quad \Psi_{Cn+1|G\bullet C2/e}$

$C_n|G\bullet C_2$

Figure 23

Note that we have chosen e such that $C_{n+1}|G\bullet C_2\backslash e$ is $A\bullet G\bullet B$ where A is a path with n vertices (2 vertices from C_{n+1} are in G, only one of which is connected to A) and B is a path with 2 vertices, one of which is connected to G. As shown above in Example 5, $\Psi_A=t(t-1)^{n-1}$ and $\Psi_B=t(t-1)$. By applying Theorem 2.1 twice, $\Psi_{Cn+1|G\bullet C2\backslash e}=(t-1)^n\Psi_G$.

$\Psi_{Cn+1|G\cdot C2}=(t-1)^n\Psi_G - \Psi_{Cn|G\cdot C2}$
$=(t-1)^n\Psi_G - (\Psi_{Cn})(\Psi_G)/t$, by our assumption.
$=(t(t-1)^n)(\Psi_G)/t - (\Psi_{Cn})(\Psi_G)/t$
$=(t(t-1)^n-\Psi_{Cn})(\Psi_G)/t$
$=(\Psi_{Cn+1})(\Psi_G)/t$, by Lemma 2.2.

By induction, $\Psi_{Cn|G\cdot C2}=(\Psi_{Cn})(\Psi_G)/t$.

In our case, if G is C_m, then $\Psi_{Cn|Cm\cdot C2}=(\Psi_{Cn})(\Psi_{Cm})/t$. Thus, for general pairs of graphs, $C_n\cdot C_m$ and $C_n|C_m\cdot C_2$, $\Psi_{cn\cdot Cm}=\Psi_{Cn|Cm\cdot C2}$.

Section 3

The next step in determining if the Orlik-Solomon algebras induced by the matroids formed by $C_n\cdot C_m$ and $C_n|C_m\cdot C_2$ are isomorphic is to find a Φ: $E(C_n\cdot C_m)\to E(C_n|C_m\cdot C_2)$ such that Φ is bijective. We look again at $C_3\cdot C_3$ and $C_3|C_3\cdot C_2$.

Figure 24

Note that the circuits of $C_3\cdot C_3$ are 123 and 456; the circuits of $C_3|C_3\cdot C_2$, 123 and 345. We will denote the basis of $E(C_3\cdot C_3)$ by $e_1, ..., e_6$ and the basis of $E(C_3|C_3\cdot C_2)$ by $f_1, ..., f_6$. We will define Φ such that

$\Phi(e_1)=f_1$	$\Phi(e_2)=f_2$	$\Phi(e_3)=f_3$
$\Phi(e_4)=f_3-f_5+f_6$	$\Phi(e_5)=f_4-f_5+f_6$	$\Phi(e_6)=f_6$

We define Φ' such that

$\Phi'(f_1)=e_1$	$\Phi'(f_2)=e_2$	$\Phi'(f_3)=e_3$
$\Phi'(f_4)=e_3-e_4+e_5$	$\Phi'(f_5)=e_3-e_4+e_6$	$\Phi'(f_6)=e_6$

Is $\Phi'=\Phi^{-1}$? It is trivial that $\Phi'(\Phi(e_i))=e_i$ for I=1,2,3, or 6. For I=4 or 5,

$\Phi'(\Phi(e_4))=\Phi'(f_3-f_5+f_6)$
$=e_3-(e_3-e_4+e_6)+e_6$
$=e_3-e_3+e_4-e_6+e_6$
$=e_4$

$\Phi'(\Phi(e_5))=\Phi'(f_4-f_5+f_6)$
$=e_3-e_4+e_5-(e_3-e_4+e_6)+e_6$
$=e_3-e_4+e_5-e_3+e_4-e_6+e_6$
$=e_5$

Thus, $\Phi'=\Phi^{-1}$, and so Φ is bijective. The remaining question is, does Φ take the circuits of $C_3 \bullet C_3$ to the circuits of $C_3|C_3 \bullet C_2$? To do this, we prove two lemmas, the first of which will be used to prove the second.

Lemma 3.1. *$e_{ij}(\partial(e_{i1i2\cdots ip}))=e_{i1\cdots ip}$ where i_j is in the sequence $i_1 \ldots i_p$*

Proof: Recall from Definition 1.4 above that $e_{ij}=-e_{ij}$ and that

$$\partial(e_{i1i2\cdots ip})=\sum_{j=1}^{p}(-1)^{j-1}e_{i1\cdots\hat{ij}\cdots ip}$$

Then, $e_{ij}(\partial(e_{i1i2\cdots ip}))=e_{ij}(e_{i2\cdots ij\cdots ip} - e_{i1i3\cdots ij\cdots ip} + \cdots + (-1)^{j-1}e_{i1\cdots\hat{ij}\cdots ip} + \cdots + (-1)^{p-1}e_{i1\cdots ij\cdots ip-1})$

$=e_{ij}(e_{i2\cdots ij\cdots ip}) - e_{ij}(e_{i1i3\cdots ij\cdots ip}) + \cdots + (-1)^{j-1}e_{ij}(e_{i1\cdots\hat{ij}\cdots ip}) + \cdots + (-1)^{p-1}e_{ij}(e_{i1\cdots ij\cdots ip-1}))$

$=(-1)^{j-1}e_{ij}(e_{i1\cdots\hat{ij}\cdots ip})$, since every other term contains $e_{ij\cdots ij}=0$.

$=(-1)^{j-1}(-1)^{1}e_{i1ij\cdots\hat{ij}\cdots ip}$, since $x\wedge y=(-1)^{(\deg(x))(\deg(y))}y\wedge x$, from Definition 1.4.

$=(-1)^{j-1}(-1)^{2}e_{i1i2ij\cdots\hat{ij}\cdots ip}$

$=(-1)^{j-1}(-1)^{j-1}e_{i1j\cdots ij-1ij\hat{ij}\cdots ip}$

$=e_{i1j\cdots ij-1ij\cdots ip}$

$=e_{i1\cdots ip}$

Corollary 3.2. *$(\partial(e_{i1i2\cdots ip}))e_{ij}=(-1)^{p-1}e_{i1\cdots ip}$*

Proof: $x\wedge y=(-1)^{(\deg(x))(\deg(y))}y\wedge x$, and $\deg(\partial(e_{i1\cdots ip}))=p-1$ (There are p terms of p-1 degree each), so
$(\partial(e_{i1i2\cdots ip}))e_{ij}=(-1)^{(p-1)(1)}\, e_{ij}(\partial(e_{i1i2\cdots ip}))$

$=(-1)^{p-1}e_{i1\cdots ip}$

Lemma 3.3. *$\partial(e_{i1i2\cdots ip})=(e_{i2}-e_{i1})(e_{i3}-e_{i2})\cdots(e_{ip}-e_{ip-1})$.*

Proof: $\partial(e_{i1i2i3})=e_{i2i3} - e_{i1i3} + e_{i1i2}$

$(e_{i2}-e_{i1})(e_{i3}-e_{i2})=e_{i2i3} - e_{i2i2} - e_{i1i3} + e_{i1i2}$

$=e_{i2i3} - e_{i1i3} + e_{i1i2}=\partial(e_{i1i2i3})$

$\partial(e_{i1i2\cdots ip})=(e_{i2}-e_{i1})(e_{i3}-e_{i2})\cdots(e_{ip}-e_{ip-1}) \rightarrow \partial(e_{i1i2\cdots ip+1})=(e_{i2}-e_{i1})(e_{i3}-e_{i2})\cdots(e_{ip+1}-e_{ip})$

Assume $\partial(e_{i1i2\cdots ip})=(e_{i2}-e_{i1})(e_{i3}-e_{i2})\cdots(e_{ip}-e_{ip-1})$.

$$\partial(e_{i1i2\cdots ip+1})=\sum_{j=1}^{p+1}(-1)^{j-1}e_{i1\cdots\hat{ij}\cdots ip+1}$$

$=e_{i2\cdots ip+1} - e_{i1i3\cdots ip+1} + e_{i1i2i4\cdots ip+1} - \cdots + (-1)^{p-1}e_{i1\cdots\hat{ij}\cdots ip}$

$=(e_{i2\cdots ip} - e_{i1i3\cdots ip} + e_{i1i2i4\cdots ip} - \cdots + (-1)^{p-1}e_{i1\cdots ij\cdots ip-1})e_{ip+1} + (-1)^{p}e_{i1\cdots ij\cdots ip}$

$=(\partial(e_{i1\cdots ip}))e_{ip+1} + (-1)^{p}(-1)^{p-1}(\partial(e_{i1\cdots ip}))e_{ip}$,

(since $(\partial(e_{i1i2\cdots ip}))e_{ij}=(-1)^{p-1}e_{i1\cdots ip}$ if i_j is in the sequence $i_1\cdots i_p$ by the corollary above.)

$=(\partial(e_{i1\cdots ip}))(e_{ip+1} - e_{ip})$

$=(e_{i2}-e_{i1})(e_{i3}-e_{i2})\cdots(e_{ip}-e_{ip-1})(e_{ip+1}-e_{ip})$, by our assumption.

By induction, $\partial(e_{i1i2\cdots ip})=(e_{i2}-e_{i1})(e_{i3}-e_{i2})(e_{ip}-e_{ip-1})$.

We will now determine if Φ, given above, will take the circuits of $C_3 \bullet C_3$ to the circuits of $C_3|C_3 \bullet C_2$. Recall that the circuits of $C_3 \bullet C_3$ are 123 and 456; the circuits of $C_3|C_3 \bullet C_2$, 123 and 345. It is natural to take e_{123} to f_{123} and e_{456} to f_{345}.

$$\begin{aligned}\Phi(\partial(e_{123})) &= \Phi[(e_2 - e_1)(e_3 - e_2)] \\ &= (f_2 - f_1)(f_3 - f_2) \\ &= \partial(f_{123})\end{aligned}$$

$$\begin{aligned}\Phi(\partial(e_{456})) &= \Phi[(e_5 - e_4)(e_6 - e_5)] \\ &= (f_4 - f_5 + f_6 - (f_3 - f_5 + f_6))(f_6 - f_4 - f_5 + f_6) \\ &= (f_4 - f_5 + f_6 - f_3 + f_5 - f_6))(f_6 - f_4 - f_5 + f_6) \\ &= (f_4 - f_3)(f_5 - f_4) \\ &= \partial(f_{345})\end{aligned}$$

Φ does, then, take the circuits of $C_3 \bullet C_3$ to the circuits of $C_3|C_3 \bullet C_2$, and Φ is bijective. Therefore, we can say that $A(C_3 \bullet C_3) \cong A(C_3|C_3 \bullet C_2)$. We will now show in a similar fashion that $A(C_n \bullet C_m) \cong A(C_n|C_m \bullet C_2)$.

Theorem 3.4. *Let B be $C_n \bullet C_m$ and C be $C_n|C_m \bullet C_2$. $A(B) \cong A(C)$.*

$C_n \bullet C_m$ $\qquad$ $C_n|C_m \bullet C_2$

Figure 25

Proof: In B, let the edges of C_n be labeled 1 through n and let the edges of C_m be labeled $n+1$ through $n+m$. The basis of $E(B)$ is $e_1, ..., e_{n+m}$. The circuits of B are $1...n$ and $n+1...n+m$.

In C, let the edges of C_n be labeled 1 through n, let the edges of C_m be labeled n through $n+m-1$, and let the edge of C_2 be labeled $n+m$. The basis of $E(C)$ is $f_1, ..., f_{n+m}$. The circuits of C are $1...n$ and $n ... n+m-1$.

Define Φ to be

$\Phi(e_i) = f_i$, $1 \le i \le n$ and $i = n+m$

$\Phi(e_i) = f_{i-1} - f_{n+m-1} + f_{n+m}$, otherwise.

We claim that Φ is bijective.

Define Φ' to be

$\Phi(f_i) = e_i$, $1 \le i \le n$ and $i = n+m$

$\Phi(f_i) = e_n - e_{n+1} + e_{i+1}$, otherwise.

It is trivial that, for $1 \leq i \leq n$ and $i=n+m$, $\Phi'(\Phi(e_i))=f_i$. For $i=n+1$,

$$
\begin{aligned}
\Phi'(\Phi(e_{n+1})) &= \Phi'(f_n - f_{n+m-1} + f_{n+m}) \\
&= e_n - (e_n - e_{n+1} + e_{n+m}) + e_{n+m} \\
&= e_n - e_n + e_{n+1} - e_{n+m} + e_{n+m} \\
&= e_{n+1}
\end{aligned}
$$

For all other i,

$$
\begin{aligned}
\Phi'(\Phi(e_i)) &= \Phi'(f_{i-1} - f_{n+m-1} + f_{n+m}) \\
&= e_n - e_{n+1} + e_i - (e_n - e_{n+1} + e_{n+m}) + e_{n+m} \\
&= e_n - e_{n+1} + e_i - e_n + e_{n+1} - e_{n+m} + e_{n+m} \\
&= e_i
\end{aligned}
$$

$\Phi' = \Phi^{-1}$, and thus, Φ is bijective. We will now determine if Φ takes the circuits of B to the circuits of C. Recall that the circuits of B are $1...n$ and $n+1...n+m$; the circuits of C, $1...n$ and $n...n+m-1$. It is natural to take $e_{1...n}$ to $f_{1...n}$ and $e_{n+1...n+m}$ to $f_{n...n+m-1}$. It is also trivial to take $e_{1...n}$ to $f_{1...n}$, so, without loss of generality, we will show only the other case.

$$
\begin{aligned}
\Phi(\partial(e_{n+1...n+m})) &= \Phi[(e_{n+2} - e_{n+1})(e_{n+3} - e_{n+2}) \cdots (e_{n+m} - e_{n+m-1})] \\
&= (f_{n+1} - f_{n+m-1} + f_{n+m} - (f_n - f_{n+m-1} + f_{n+m}))(f_{n+2} - f_{n+m-1} + f_{n+m} - (f_{n+1} - f_{n+m-1} + f_{n+m})) \cdots (f_{n+m} - (f_{n+m-2} - f_{n+m-1} + f_{n+m})) \\
&= (f_{n+1} - f_{n+m-1} + f_{n+m} - f_n + f_{n+m-1} - f_{n+m})(f_{n+2} - f_{n+m-1} + f_{n+m} - f_{n+1} + f_{n+m-1} - f_{n+m}) \cdots (f_{n+m} - f_{n+m-2} + f_{n+m-1} - f_{n+m}) \\
&= (f_{n+1} - f_n)(f_{n+2} - f_{n+1}) \cdots (f_{n+m-1} - f_{n+m-2}) \\
&= \partial(f_{n...n+m-1})
\end{aligned}
$$

Φ is bijective, and takes the circuits of B to the circuits of C. Therefore, $A(B) \cong A(C)$.
B is $C_n \bullet C_m$ and C is $C_n | C_m \bullet C_2$, so $A(C_n \bullet C_m) \cong A(C_n | C_m \bullet C_2)$.

Section 4

We have established that, in the general case of $C_n \bullet C_m$ and $C_n | C_m \bullet C_2$, $A(C_n \bullet C_m) \cong A(C_n | C_m \bullet C_2)$. It is left to determine if $T_{Cn\bullet Cm} = T_{Cn\bullet Cm|C2}$. Again, we look at the case of $C_3 \bullet C_3$ and $C_3 | C_3 \bullet C_2$.

We have shown that $T_{C3\bullet C3} = (T_{C3})(T_{C3}) = (x^2+x+y)(x^2+x+y) = x^4+2x^3+2x^2y+x^2+2xy+y^2$. Look at $C_3 | C_3 \bullet C_2$.

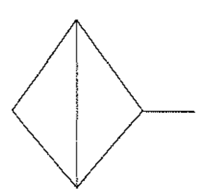


Figure 26

In applying deletion-contraction, choose e to be the isthmus. Then $T_{C3 \cdot C3|C2} = x(j)$, where j is some polynomial in x and y. Note that the expansion of $T_{C3 \cdot C3}$ contains y^2 as a term. Regardless of the value of j, $T_{C3 \cdot C3|C2}$ will never contain a y^2 term; all terms will have x as a factor. Therefore, $T_{C3 \cdot C3} \neq T_{C3 \cdot C3|C2}$.

Having one counterexample, we can now conclude that given matroids M_1 and M_2, $A(M_1) \cong A(M_2)$ does not imply that the Tutte polynomials are equal. However, we will now show that in the general case of $C_n \cdot C_m$ and $C_n|C_m \cdot C_2$, $T_{Cn \cdot Cm} \neq T_{Cn \cdot Cm|C2}$.

We know that $T_{C3 \cdot C3} = (T_{C3})(T_{C3 \cdot})$. We will now prove that in the general case of $C_n \cdot C_m$ that $T_{Cn \cdot Cm} = (T_{Cn})(T_{Cm})$.

Lemma 4.1. $T_{Cn} = \sum_{j=1}^{n-1} (x^j) + y \;\forall\; n \geq 3.$

Proof:

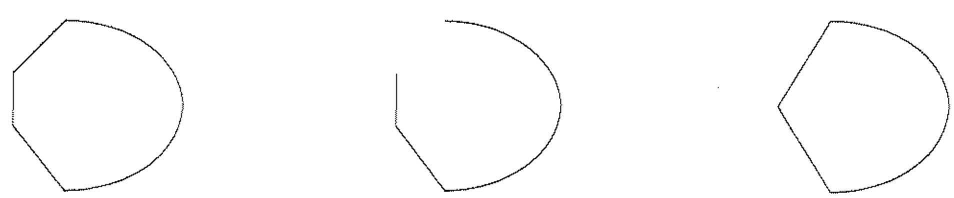
$T_{C3} = x^2 + x + y$ by Example 2 above.

$$\sum_{j=1}^{3-1} (x^j) + y = \sum_{j=1}^{2} (x^j) + y$$

$$= x^2 + x + y = T_{C3}$$

$$T_{Cn} = \sum_{j=1}^{n-1} (x^j) + y \rightarrow T_{Cn+1} = \sum_{j=1}^{n} (x^j) + y$$

Assume $T_{Cn} = \sum_{j=1}^{n-1} (x^j) + y$. By deletion-contraction,



T_{Cn+1} = $T_{Cn+1\backslash e}$ (n isthmuses) + $T_{Cn+1/e}$ (C_n)

Figure 27

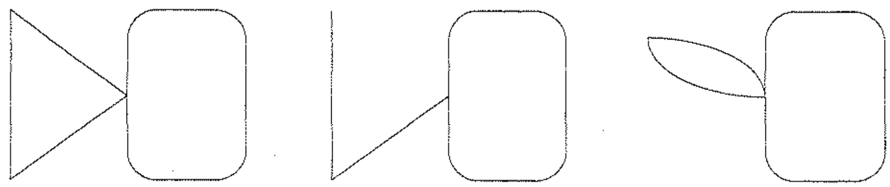
$$T_{Cn+1} = x^n + T_{Cn}$$

$$T_{Cn+1} = x^n + \sum_{j=1}^{n-1} (x^j) + y, \text{ by our assumption.}$$

$$= \sum_{j=1}^{n} (x^j) + y$$

By induction, $T_{Cn} = \sum_{j=1}^{n-1} (x^j) + y \;\forall n \geq 3.$

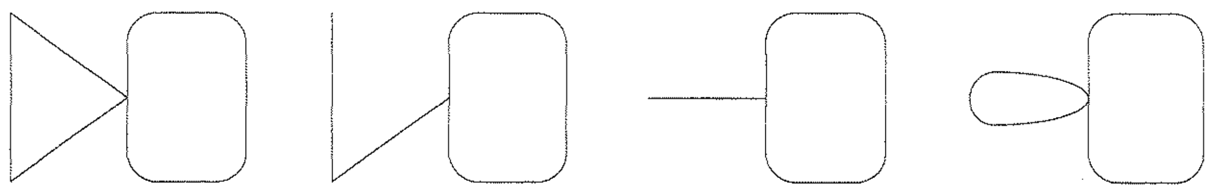
Theorem 4.2. $T_{C_n \bullet G} = (T_{C_n})(T_G)$.



$T_{C3 \bullet G} \quad = \quad T_{C3 \bullet G \backslash e} \quad + \quad T_{C3 \bullet G / e}$

Figure 28

Proof: Applying *T*1 and *T*2 twice, $T_{C3 \bullet G \backslash e} = x^2(T_G)$. We let A be the graph $C_3 \bullet G/e$.



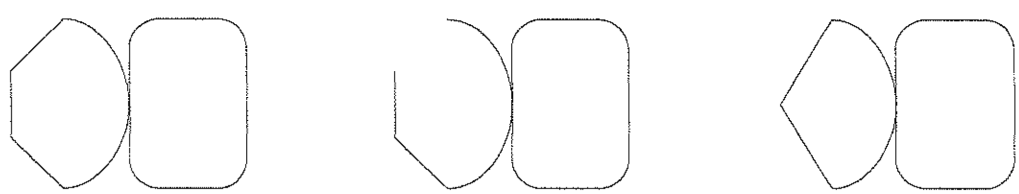
$T_{C3 \bullet G} \quad = \quad x^2(T_G) \quad + \quad T_{A \backslash e} \quad + \quad T_{A/e}$

Figure 29

Applying *T*1 and *T*2 to $A \backslash e$ $T_{A \backslash e} = x(T_G)$. Applying *T*1 and *T*2 to A/e $T_{A \backslash e} = y(T_G)$. Therefore $T_{C3 \bullet G} = (x^2 + x + y)T_G = (T_{C3})(T_G)$.

$T_{Cn \bullet G} = (T_{Cn})(T_G) \rightarrow T_{Cn+1 \bullet G} = (T_{Cn+1})(T_G)$

Assume $T_{Cn \bullet G} = (T_{Cn})(T_G)$. By deletion-contraction,



$T_{Cn+1 \bullet G} \quad = \quad T_{Cn+1 \bullet G \backslash e} \quad + \quad T_{Cn+1 \bullet G/e}$

$C_{n+1} \bullet G \backslash e$ has n isthmuses $\qquad C_n \bullet G$

Figure 30

Applying *T*1 and *T*2 repeatedly, $T_{Cn+1 \bullet G \backslash e} = x^n(T_G)$. Thus, $T_{Cn+1 \bullet G} = (x^n + T_{Cn})(T_G) = (T_{Cn+1})(T_G)$, by the above lemma and our assumption.

Therefore, by induction, $T_{Cn \bullet G} = (T_{Cn})(T_G)$ $\forall$ $n \geq 3$.

Letting G be C_m, $T_{Cn \bullet Cm} = (T_{cn})(T_{Cm})$.

Applying the same logic used above to the case of $C_3 \bullet C_3$ and $C_3|C_3 \bullet C_2$, look at $C_n|C_m \bullet C_2$.

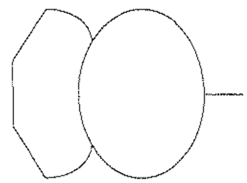


Figure 31

$T_{Cn|Cm \bullet 2} = x(j)$ where j is some polynomial in x and y. $T_{Cn \bullet Cm} = (T_{Cn})(T_{Cm})$. In the expansion of $(T_{Cn})(T_{Cm})$, there will be a term y^2, since both (T_{Cn}) and (T_{Cm}) have a term y. $T_{Cn|Cm \bullet 2}$ will have no term with only y as factors; all terms will have x as a factor. Therefore, $T_{Cn \bullet Cm} \neq T_{Cn|Cm \bullet 2}$.

Since we have $n, m \in \mathbb{N}$ and $n, m \geq 3$, we have an infinite family of counterexamples with which to conclude that given matroids M_1 and M_2, $A(M_1) \cong A(M_2)$ does not imply that the Tutte polynomials are equal.

Questions of the existence of two-connected graphs (connected graphs for which given G, $\forall e \in E(G)$, $G \backslash e$ is connected) which have isomorphic Orlik-Solomon algebras and non-isomorphic Tutte polynomials will be considered in the future. Currently, graphs of the form $C_n|C_4|C_m$ and $C_n|C_m|C_4$ are being examined.

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