Regular Strictly Bi-Transitive Graphs

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Introduction:

The *degree* of a vertex v in the graph Γ is the number of edges incident with v. Now, the minimum degree among the vertices of Γ is given by min deg Γ and the maximum degree among the vertices of Γ is given by max deg Γ. If min deg Γ = max deg Γ = r, then all the vertices have the same degree and G is called *regular* of degree r.

Now, if Γ is a graph, then a *symmetry* or *automorphism* of Γ is a permutation σ on its vertices so that for every edge (x,y), then $(x\sigma, y\sigma) = (x,y)\sigma$ is an edge. Thus each automorphism of Γ is a one-to-one and onto mapping of the vertices of Γ which preserves adjacency. It is obvious that any automorphism sends any vertex onto another vertex of the same degree. Furthermore, let $G = G(\Gamma)$ be the collection of all symmetries of Γ. It follows that $G(\Gamma)$ is a group under right composition.

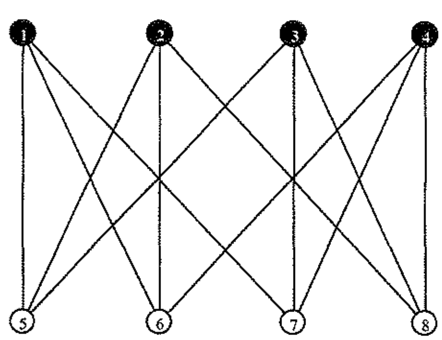
A graph Γ is said to be *vertex-transitive* if given any two vertices of Γ, v_1 and v_2, then there is some automorphism α of $G(\Gamma)$ such that $v_1\alpha = v_2$. Similarly, a graph Γ is said to be *edge-transitive* if given any two edges of Γ, u_1 and u_2, then there is some automorphism β of $G(\Gamma)$ such that $u_1\beta = u_2$.

Consider the edge of a graph Γ, (v_1, v_2). This edge can be directed in two ways: (v_1, v_2) or (v_2, v_1). Each of these representations of the edge is called a *dart*. Note that with each edge of a graph, there are two darts associated with it. Furthermore, a graph Γ is said to be *dart-transitive* provided that for any two darts of Γ, (x_1, x_2) and (y_1, y_2) there is some automorphism γ of $G(\Gamma)$ such that $(x_1, x_2)\gamma = (y_1, y_2)$.

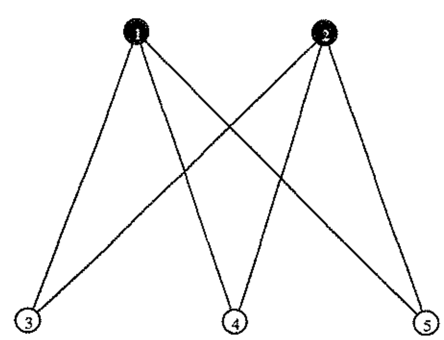
A graph Γ is *bipartite* provided that the vertex set $V(\Gamma)$ can be partitioned into two nonempty subsets V_1 and V_2 and provided that every edge of Γ joins two vertices, one from V_1 and the other from V_2. If there exists an edge in Γ which joins every vertex of V_1 to every vertex of V_2, then Γ is said to be a *complete bipartite* graph. A complete bipartite graph will be denoted by $K_{x,y}$ where x is the number of top vertices and y is the number of bottom vertices. It follows clearly from the definition that the top vertices will be of degree y and the bottom vertices will be of degree x.

Strictly Bi-Transitive Graphs

Γ is said to be *bi-transitive* provided that Γ is bipartite and edge-transitive and $G(\Gamma)$ has a subgroup $G^+(\Gamma)$ which preserves color. In other words, if the graph Γ is partitioned into black and white vertices, then the subgroup $G^+(\Gamma)$ contains automorphisms which transitively maps white vertices onto white vertices and transitively maps black vertices onto black vertices, but does not map white vertices onto black vertices or black vertices onto white vertices. Furthermore, if $G^+(\Gamma) = G(\Gamma)$, then Γ is said to be *strictly bi-transitive*. In other words, a strictly bi-transitive graph is one which is bipartite and edge-transitive, but is not vertex-transitive. Here are two examples, Γ_1 being bi-transitive and Γ_2 being strictly bi-transitive.



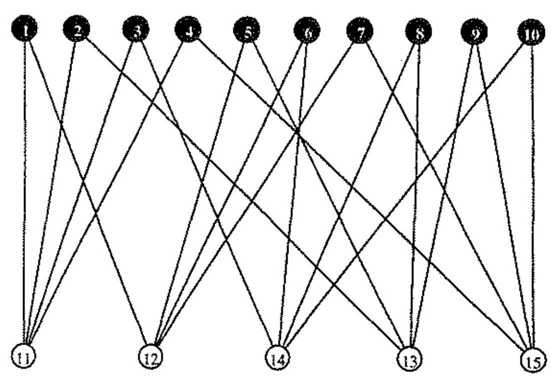
$\Gamma_1 = P(4,1,3)$



$\Gamma_2 = K_{2,3}$

In the captions of the graphs above, we see some new notation. A graph denoted by P(N, a, b) where a < b < N is called a *power set graph*, a bipartite graph whose top vertices correspond to the subsets of length b of the power set of the set {1, 2, ..., N} and whose bottom vertices correspond to the subsets of length a of the power set of the set {1, 2, ..., N}. Note that there are C(N, b) top vertices and that there are C(N, a) bottom vertices. A vertex of length b, b_1, is adjacent to a vertex of length a, a_1, provided that a_1 is a a subset of b_1. Note furthermore that the top vertices are of degree C(b, a) and the bottom vertices are of degree C(N-a, b-a).

As we saw above in the example, P(4,1,3) is bi-transitive. In a general case for the power set graphs, we can conclude that, in general, the power set graph is bi-transitive. When a + b ≠ N, then the power set graph is strictly bi-transitive. Let us look at a simple example of this.



P(5,1,2)

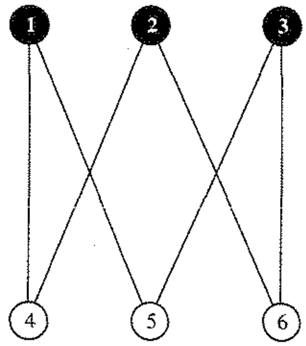
As we can see from the graph above, there can be no symmetry which maps a white vertex onto a black (or vice versa) because the degrees of the black and white vertices are different. This means that the automorphism group does not act transitively on the vertices and it follows that the graph is strictly bi-transitive. (See Appendix A for a table of properties for the power set graphs.)

It is evident from Appendix A that the power set graphs become very large for even small values of N, a, and b. Because it would be time-consuming and tedious to draw these graphs by hand, it became necessary to generate these graphs using output files from a *Mathematica* program which I wrote and the ability of *Groups and Graphs* to read in these files and draw the resulting graph. (To view the algorithm and *Mathematica* code, please refer to Appendix B.)

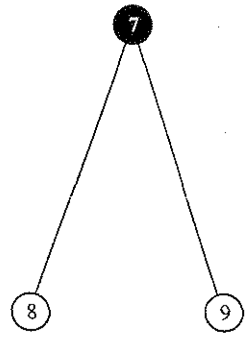
Another example of a strictly bi-transitive graph is the complete bipartite graph $K_{x,y}$ where $x \neq y$. We saw an example of this above with $K_{2,3}$. By definition of the complete bipartite graph, the degree of the black vertices will be different from the degree of the white vertices if $x \neq y$. This forces the graph to be strictly bi-transitive because the automorphism group does not act transitively on the vertices. In the case where $x = y$, we find that all vertices have the same degree and furthemore, there exists a symmetry which maps a black vertex onto a white vertex. Because the graph $K_{x,y}$ where $x = y$ is vertex-transitive, it follows that the graph is simply bi-transitive.

Wedge Products

Several binary operations can be defined on graphs, one of which is called the wedge product. Suppose that Γ_1 and Γ_2 are graphs. The *wedge product* of Γ_1 and Γ_2, denoted $\Gamma_1 \wedge \Gamma_2$, has vertices (u_1, u_2) where u_1 is a vertex of Γ_1 and u_2 is a vertex of Γ_2. Furthermore, $(u_1, u_2)(v_1, v_2)$ is an edge of $\Gamma_1 \wedge \Gamma_2$ if $u_1 v_1$ is an edge of Γ_1 and $u_2 v_2$ is an edge of Γ_2. For example, suppose that we want to find the wedge product of P(3,1,2) and $K_{1,2}$. Note that both graphs are bipartite graphs, and so each graphs has a set of black vertices and a set of white vertices. Listed below the two graphs is an adjacency table of vertices of the wedge product and below this list we find the graph of the wedge product of the two graphs.



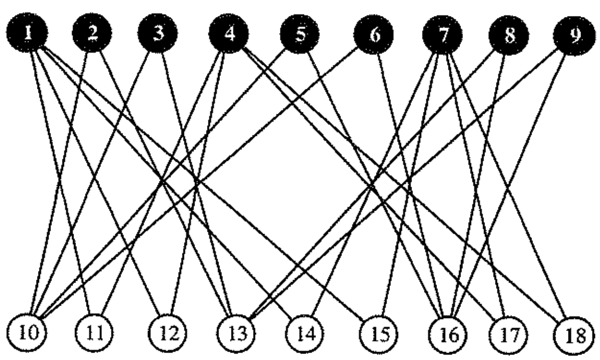
P(3,1,2)



$K_{1,2}$

Vertices of P(3,1,2) ∧ $K_{1,2}$ and adjacent vertices

1 - (1,7)	11, 12, 14 , 15		10 - (4,7)	2, 3, 5, 6
2 - (1,8)	10, 13		11 - (4,8)	1, 4
3 - (1,9)	10, 13		12 - (4,9)	1, 4
4 - (2,7)	11, 12, 17, 18		13 - (5,7)	2, 3, 8, 9
5 - (2,8)	10, 16		14 - (5,8)	1, 7
6 - (2,9)	10, 16		15 - (5,9)	1, 7
7 - (3,7)	14, 15, 17, 18		16 - (6,7)	5, 6, 8, 9
8 - (3,8)	13, 16		17 - (6,8)	4, 7
9 - (3,9)	13, 16		18 - (6,9)	4, 7

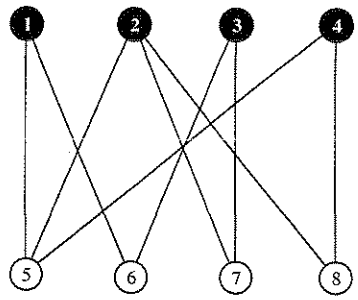


P(3,1,2) ∧ $K_{1,2}$

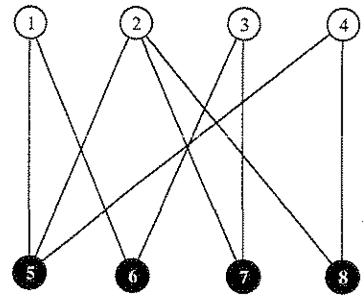
Note that if Γ_1 has m vertices and Γ_2 has n vertices, then $\Gamma_1 \wedge \Gamma_2$ will have mn vertices. Note also that if u_1 is vertex of Γ_1 with degree d_1 and u_2 is a vertex of Γ_2 of degree d_2, then the vertex (u_1,u_2) of $\Gamma_1 \wedge \Gamma_2$ will have degree $d_1 d_2$.

If Γ_1 and Γ_2 are both bi-transitive graphs , then both Γ_1 and Γ_2 are bipartite. Therefore, the vertices of each graph can be partitioned into two sets, the black vertices and the white vertices. Given this, we find that in $\Gamma_1 \wedge \Gamma_2$, there are four kinds of vertices: (B,B), (B,W), (W,B), (W,W). By definition of the wedge product, we know that vertices (a,b) and (c,d) are adjacent in $\Gamma_1 \wedge \Gamma_2$ if a is adjacent to c in Γ_1 and b is adjacent to d in Γ_2. Because Γ_1 and Γ_2 are bi-partite, we have that the black vertices in each graph are adjacent to white vertices only. This gives that a (B,B) vertex in $\Gamma_1 \wedge \Gamma_2$ can only be adjacent to a (W,W) vertex of $\Gamma_1 \wedge \Gamma_2$ and vice versa. Similarly, we find that (B,W) vertices can only be adjacent to (W,B) vertices in $\Gamma_1 \wedge \Gamma_2$ and vice versa. This furthermore gives that the (B,B) and (W,W) vertices and the edges joining them is disjoint to the (B,W) and (W,B) vertices and the edges joining them.

When considering the graph $\Gamma_1 \wedge \Gamma_2$, it is suffice to consider $\Gamma$where Γ is the part of $\Gamma_1 \wedge \Gamma_2$ consisting of the (B,B) and (W,W) vertices and the edges joining them for the following reason: Properties of a graph still hold regardless of our choice of black and white vertices. For instance, consider the following graphs Γ_1 and Γ_1'. We see that these graphs are the same graphs except for the coloring of their vertices.



Γ_1



Γ_1'

If Γ_1 and Γ_1' were wedged with the same graph Γ_2, we see that the (B,B), (B,W), (W,B), and (W,W) vertices of $\Gamma_1 \wedge \Gamma_2$ would respectively be the same corresponding (W,B), (W,W), (B,B), (B,W) vertices of $\Gamma_1' \wedge \Gamma_2$. Furthermore, the Γ of $\Gamma_1 \wedge \Gamma_2$ is the same as the part of $\Gamma_1' \wedge \Gamma_2$ consisiting of the (W,B) and (B,W) vertices and edges joining them. Therefore, it is enough to consider Γ only of any wedge product of two bi-partite graphs. This result will be useful, in particular, in the next two results.

LEMMA: If Γ_1 and Γ_2 are bi-transitive, then each compopnent of $\Gamma_1 \wedge \Gamma_2$ is bi-transitive.

Proof: In order to show that $\Gamma_1 \wedge \Gamma_2$ is bi-transitive, we must show that $\Gamma_1 \wedge \Gamma_2$ is both bipartite and edge transitive.

Γ consists of vertices the (B,B) and (W,W) vertices and edges joining them of $\Gamma_1 \wedge \Gamma_2$, as described above.

Suppose $V_1 = \{(b_1, b_2) : b_1 \in B_1 \in \Gamma_1$ and $b_2 \in B_2 \in \Gamma_2\}$ and

$V_2 = \{(w_1, w_2) : w_1 \in W_1 \in \Gamma_1$ and $w_2 \in W_2 \in \Gamma_2\}$.

If Γ is bipartite, then $V_1 \cap V_2 = \varnothing$ and there is no edge joining any two vertices of V_1 and similarly V_2. Suppose that (b_1, b_2) and (b_3, b_4) are any two vertices in V_1. If they are adjacent, then there is an edge joining them. But, by definition of $\Gamma_1 \wedge \Gamma_2$ we know that there is an edge joining (b_1, b_2) and (b_3, b_4) if and only if b_1 is adjacent to b_3 in Γ_1 and b_2 is adjacent to b_4 in Γ_2. It is obvious that there can be no edge joining b_1

to b_3 or b_2 to b_4 because Γ_1 and Γ_2 are both themselves bipartite. SImilarly, it can be shown that there are no edges joining any two vertices in V_2. Therefore, Γ is bipartite.

Since Γ_1 is edge transitive, there exists some $\sigma \in G(\Gamma_1)$ such that $(b_1 w_1)\sigma = (b_3 w_3) = ((b_1\sigma)(w_1\sigma))$. Similarly, since Γ_2 is edge-transitive, then there exists $\tau \in G(\Gamma_2)$ such that $(b_2 w_2)\tau = (b_4 w_4) = ((b_2\tau)(w_2\tau))$.

Suppose that $((b_1,b_2)(w_1,w_2))$ and $((b_3,b_4)(w_3,w_4))$ are edges in Γ. If Γ is to be edge-transitive, then there is some symmetry $\rho \in G(\Gamma)$ such that $((b_1,b_2)(w_1,w_2))\rho = ((b_3,b_4)(w_3,w_4))$.

Define ρ on Γ by $((x_1,x_2)(y_1,y_2))\rho = ((x_1\sigma)(x_2\tau))((y_1\sigma)(y_2\tau))$ or in other words $((x_1\sigma)(y_1\sigma))$is an edge in Γ_1 and $((x_2\tau)(y_2\tau))$ is an edge in Γ_2. It must now be shown that ρ is a symmetry of Γ. Suppose $((b_1,b_2)(w_1,w_2)) \in \Gamma$. Then $(b_1 w_1) \in \Gamma_1$ and $(b_2 w_2) \in \Gamma_2$. Since Γ_1 and Γ_2 are both edge-transitive, $((b_1\sigma)(w_1\sigma)) \in \Gamma_1$ and $((b_2\tau)(w_2\tau)) \in \Gamma_2$. It follows that $(((b_1\sigma),(b_2\tau))((w_1\sigma),(w_2\tau))) \in \Gamma$ which means that $((b_1,b_2)(w_1,w_2))\rho \in \Gamma$. Because ρ is a symmetry, we know that Γ is edge-transitive. Because Γ is bipartite and edge-transitive, Γ is bi-tranisitive. Therefore, $\Gamma_1 \wedge \Gamma_2$ is bi-transitive.é

This important fact tells us that because the power set graphs and the complete bipartite graphs are both bi-transitive graphs, then each part of the wedge product of a power set graph with a complete bipartite is indeed bi-transitive. Suppose that we take the wedge product of a power set graph and a complete bipartite graph such that the degree of the (B,B) vertices equals the degree of the (W,W) vertices. (See Appendix B for the algorithm and Mathematica code which generated these wedge products. See Appendix C for a table of the properties of these wedge products.) Surprisingly, we find that all of these wedge products are indeed not vertex transitive, and hence strictly bi-transitive, despite the fact that we have a regular graph. More generally, we find that wedge product of a graph with the same properties as the power set graph with a complete bipartite graph where $x \neq y$ is strictly bi-transitive:

THEOREM: Given two graphs, Γ_1 and $K_{x,y}$ where Γ_1 is a bi-transitive graph with *r* black vertices of degree *p* and *s* white vertice of degree *q*, and where $K_{x,y}$ is a complete bipartite graph in which $x \neq y$. Γ_1 is also a graph such that no pairs of vertices are adjacent to exactly the same vertices. Then $\Gamma_1 \wedge K_{x,y}$ is strictly bi-transitive.

Proof:

Since $x \neq y$, we get that $K_{x,y}$ is strictly bi-transitive and by definition, $K_{x,y}$ is bipartite and edge transitive. Now, because Γ_1 is bi-transitive, Γ_1 is bipartite and edge-transitive. By a previous lemma, it is true that each part of $\Gamma_1 \wedge K_{x,y}$ is bi-transitive (bipartite and edge-transitive). In order to show that $\Gamma_1 \wedge K_{x,y}$ is strictly bi-transitive, it must be shown that each part of $\Gamma_1 \wedge K_{x,y}$ is not vertex transitive. By the preliminary arguments, this can be done by showing that Γ(the (B,B) and (W,W) vertices of $\Gamma_1 \wedge K_{x,y}$ and the edges joining them) is not vertex transitive.

There are two cases which must be considered: when the degree of the (B,B) vertices of $\Gamma_1 \wedge K_{x,y}$ is equal to the degree of the (W,W) vertices of $\Gamma_1 \wedge K_{x,y}$ and when the degree of the (B,B) vertices of $\Gamma_1 \wedge K_{x,y}$ is less than the degree of the (W,W) vertices of $\Gamma_1 \wedge K_{x,y}$. (This case also takes care of the case when the degree of the (B,B) vertices of $\Gamma_1 \wedge K_{x,y}$ is greater than the degree of the (W,W) vertices of $\Gamma_1 \wedge K_{x,y}$ because if this were the case, then we could switch the color orientation of both graphs without changing the wedge product in any way.) By the same reasoning, without loss of generality, it can be assumed that there are *x* black vertices of degree *y* and *y* white vertices of degree *x* in $K_{x,y}$ where $x < y$.

<u>Case 1</u>: Suppose that in $\Gamma_1 \wedge K_{x,y}$, the degree of the (B,B) vertices is less than the degree of the (W,W) vertices. Since the degrees of the (B,B) and (W,W) vertices are different, there is no automorphism of Γ which sends a (B, B) vertex to a (W,W) vertex. It can therefore be concluded that Γ is not vertex-transitive, which forces Γ to be strictly bi-transitive.

Case 2: Suppose now that in $\Gamma_1 \wedge K_{x,y}$, the degree of the (B,B) vertices is equal to the degree of the (W,W) vertices. It is true that there is at least one black vertex $b_1' \in K_{x,y}$. (Primes will be used to denote a vertex from $K_{x,y}$.) So, suppose that (b_1, b_1') is a (B,B) vertex of Γ, where $b_1 \in \Gamma_1$ with degree p and $b_1' \in K_{x,y}$ with degree y. Now suppose $w_1, ..., w_p$ are the vertices in Γ_1 to which b_1 is adjacent, and suppose $w_1', ..., w_y'$ are the vertices in $K_{x,y}$ to which b_1' is adjacent. Now, (b_1, b_1') is adjacent to the py vertices (w_i, w_j') where $1 \leq i \leq p$ and $1 \leq j \leq y$.

It is true by hypothesis that there are no other black vertices in Γ_1 which are adjacent to exactly the same white vertices to which b_1 is adjacent. But in $K_{x,y}$, if $x > 1$, there is some other vertex $b_2' \in K_{x,y}$ which is also adjacent to the vertices $w_1', ..., w_y'$. Similarly, for each vertex b_m' where $m \leq x$, b_m' is adjacent to $w_1', ..., w_y'$. So in Γ, the vertices (b_1, b_m') where $1 \leq m \leq x$ are adjacent to the same py vertices (w_i, w_j') where $1 \leq i \leq p$ and $1 \leq j \leq y$.

It follows from this result that the (B,B) vertices are partitioned by x. Each set of vertices in the partition contains the x (B,B) vertices which are adjacent to exactly the same py (W,W) vertices. There are no other (B,B) vertices outside of a set of vertices $b_{k,1}, ..., b_{k,x}$ which are adjacent to exactly the same (W,W) vertices to which $b_{k,1}, ..., b_{k,x}$ are adjacent. It follows that the (B,B) vertices of Γ are partioned into the following sets (where $b_{1,1}$ is short for the vertex (b_1, b_1')) :

$b_{1,1}, ..., b_{1,x}$

$b_{2,1}, ..., b_{2,x}$

...

$b_{r,1}, ..., b_{r,x}$

Since $x < y$ and there is at least one black vertex of $K_{x,y}$, it follows that there are at least two white vertices in $K_{x,y}$. Consider the (W,W) vertex (w_1, w_1') in Γ. Then $w_1 \in \Gamma_1$ has degree q and $w_1' \in K_{x,y}$

has degree x. Now suppose that $b_1, ..., b_q$ be the vertices in Γ_1 to which w_1 is adjacent and suppose that $b_1', ..., b_x'$ are the vertices in $K_{x,y}$ to which w_1' is adjacent. Then (w_1, w_1') is adjacent to the qx vertices (b_i, b_j') where $1 \leq i \leq q$ and $1 \leq j \leq x$. It is true by hypothesis that there are no other white vertices in Γ_1 which are adjacent to exactly the same black vertices to which w_1 is adjacent . But in $K_{x,y}$, since $y \geq 2$, there is some other vertex w_2' which is also adjacent to $b_1', ..., b_x'$. Similarly, for each vertex w_n', where $n \leq y$, w_n' is adjacent to exactly $b_1', ..., b_x'$. So in , the vertices (w_1, w_n') where $1 \leq n \leq y$ are adjacent to the same qx vertices (b_i, b_j') where $1 \leq i \leq q$ and $1 \leq j \leq x$.

It follows that the (W, W) vertices are partitioned by y. Similar to the (B,B) vertex partition, there are no other (W,W) vertices outside of the set of vertices $w_{d,1}, ... ,w_{d,y}$ which are adjacent to exactly the same qx (B,B) vertices to which $w_{d,1}, ... ,w_{d,y}$ are adjacent. Furthermore, the (W,W) vertices of Γ are partitioned into the following sets:

$w_{1,1}, ... ,w_{1,y}$

$w_{2,1}, ... ,w_{2,y}$

...

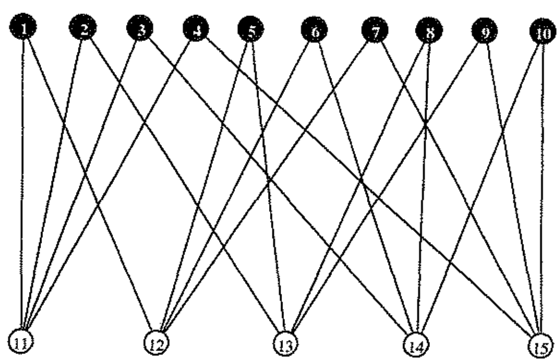
$w_{s,1}, ... ,w_{s,y}$.

Now, because $x < y$, we have that Γ is not vertex-transitive by the following reasoning: Suppose that Γ is vertex-transitive, then there is some $\sigma \in G(\Gamma)$ such that $b_{1,1}\sigma = w_{1,1}$. Now, $w_{1,1}\sigma$ could be any one of $b_{1,1}, ... , b_{1,x}, b_{2,1}, ... , b_{2,x}, ... , b_{r,1}, ... , b_{r,x}$. Suppose that, without loss of generality, $w_{1,1}\sigma = b_{n,1}$ where $1 \leq n \leq r$. Now, $w_{1,1}\sigma, ... , w_{1,y}\sigma$ must be adjacent to exactly the same $qx = py$ vertices. So each of $w_{1,2}\sigma, ... , w_{1,y}\sigma$ must be one of $b_{n,2}, ... , b_{n,x}$. Since $x < y$, it is true for some $1 \leq i,j \leq y$ and $1 \leq k \leq x$, that $w_{1,i}\sigma = b_{n,k} = w_{1,j}\sigma$. If this is true, then σ would not be one-to-one, and furthermore, σ is not an automorphism in $G(\Gamma)$. Since there is no symmetry $\sigma \in G(\Gamma)$ which takes a (B,B) vertex to a (W,W) vertex, the graph Γ is not vertex-transitive and by definition Γ is strictly bi-transitive. é

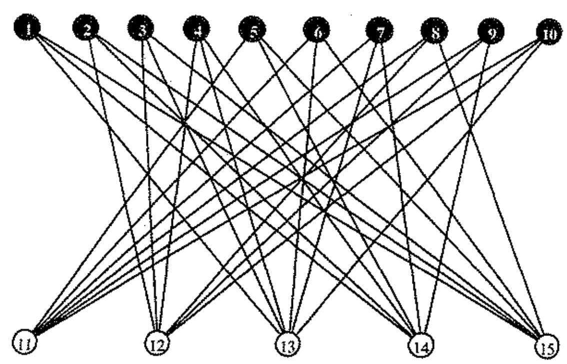
DISTANCE-3 GRAPHS:

The *distance*, d(u,v), between two vertices u and v of a graph Γ, is the length of shortest path joining u and v if such a path exists. Note that d(u,v) = d(v,u).

A *distance-3 graph*, denoted Distance3(Γ) where Γ is a graph, is a new graph with the vertex set of Γ. A vertex in Γ is adjacent in the new graph, Distance3(Γ), to all of the vertices at a distance 3 from it in Γ. For example, compare P(5,1,2) to Distance3(P(5,1,2).



P(5,1,2)



Distance3(P(5,1,2))

Note in the previous example, any black (white) vertex in Distance3(P(5,1,2)) is adjacent to exactly those white (black) vertices to which it is not adjacent in P(5,1,2). Realizing this, the following generalization can be made:

LEMMA: In the graph P(N,1,b), a white (black) vertex will be at a distance 1 or 3 from any of the black (white) vertices. (See the last column of summary table in Appendix A)

Proof:

Since P(N,1,b) is a bipartite graph, a black (white) vertex is at an odd distance from all of the white (black) vertices. Without loss of generality, suppose that the white vertices are those corresponding to the singleton sets. Now there are N white vertices. Because P(N,a,b) for any a, b $\in \mathbb{Z}^+$ such that a < b < N is bi-transitive, it follows clearly that P(N,1,b) is bi-transitive. Furthermore, P(N,1,b) is transitive among the black vertices and similarly the white vertices. So, without loss of generality, consider the white vertex corresponding to the singleton set {1}. By definition of P(N,1,b), the vertex corresponding to the singleton set {1} will be adjacent to (at a distance 1 from) the C(N-1,b-1) vertices corresponding to the sets of length *b* containing the element *1*. Among these sets of length *b* containing the element *1* are the following sets: {1, ..., b-1, b}, {1, ..., b-1, b+1}, ..., {1, ..., b-1, N}. Since, among these sets are all other elements of the set {1, ..., N} besides the element 1, then the vertices corresponding to these sets collectively are adjacent to the vertices corresponding to all the other singleton sets {2}, ..., {N}.

Now consider the C(N-1,b) black vertices to which the vertex corresponding to the singleton set {1} is not adjacent. These are the following sets: {2, ..., b+1}, {2, ..., b, b+2}, ..., {2, ..., b,N}, {2, 4, ...,b+2}, ..., {2, 4, ..., b+1, N}, ..., {2, (N-b)+2, ..., N}, ..., {(N-b)+1, ..., N}. Now because all of the vertices corresponding to these sets are adjacent to at least oen of the vertices corresponding to the singleton sets {2}, ..., {N}, it follows that the vertices corresponding to these sets are at a distance 3 from the vertex corresponding to the set {1}.

Because C(N-1,b) + C(N-1,b-1) = C(N,b), it follows that the vertex corresponding to the set {1} is at a distance one or three from the C(N,b) vertices corresponding to the sets of length b. It follows that any white vertex is at a distance one or three from any of the black vertices. Because d(u,v) = d(v,u), it follows that any black vertex of P(N,1,b) is at a distance one or three from any of the white vertices. é

The Distance3(P(N,1,b)) turns out to be a strictly bi-transitive graph, but note from the table in Appendix D the similarities between the graphs P(N,1,b) and the graph Distance3(P(N,1,N-b)). This observation leads to the next theorem that, in fact, P(N,1,b) ≅ Distance3(P(N,1,N-b)).

THEOREM: $P(N,1,b) \cong$ Distance3(P(N,1,N-b)).

Proof:

Two graphs are isomorphic if there is a one-to-one correspondence between the vertices in the two graphs such that a pair of vertices are adjacent in one graph if and only if the corresponding pair of vertices are adjacent in the other graph (Tucker).

As in the previous proof, in any power set graph of the form P(N,1,b), a black (whtie) vertex is at a distance 1 or 3 from any other the white (black) vertices. Furthermore, using the convention that the black vertices are those corresponding to the sets of length b and the white vertices are those corresponding to the singleton sets, it follows that the black vertices are of degree b and the white vertices are of degree C(N-1,b-1).

Consider the graph P(N,1,N-b). The black vertices are of degree N-b, and the white vertices are of degree C(N-1,(N-b)-1). SInce black (white) vertices are at a distance 1 or 3 from any of the white(black) vertices, int he graph Distance3(P(N,1,N-b)), the black (white) vertices will be adjacent to those white (black) vertices which they are not adjacent to in P(N,1,N-b). So the black vertices of Distance3(P(N,1,N-b)) will have degree N-(N-b) = b and the white vertices will have degree C(N,N-b) - C(N-1,(N-b)-1) = C(N-1,N-b) = C(N-1,b-1). SO in both graphs P(N,1,b) and Distance3(P(N,1,N-b)), there are C(N,b) = C(N,N-b) black vertices of degree b and there are N white vertices of degree C(N-1,b-1).

Let Φ: V(P(N,1,b)) $\rightarrow$ V(Distance3(P(N,1,N-b))) take the white vertices of P(N,1,b) to the singelton sets to the white vertices corresponding to the same singleton set in Distance3(P(N,1,N-b)). In other words, if v({a}) means the vertex corresponding to the singleton set {a}, then under Φ : v({1}) $\rightarrow$ v({1}), ..., v({N}) $\rightarrow$ v({N}).

Now, in P(N,1,b), a white vertex corresponding to a singleton set is adjacent to a black vertex if the singleton set is a subset of the set to which the black vertex corresponds and in Distance3(P(N,1,N-b)) a white vertex corresponding to a singleton set is adjacent toa black vertex if the singleton set if the singleton set is NOT a subset of the set to which the black vertex corresponds.

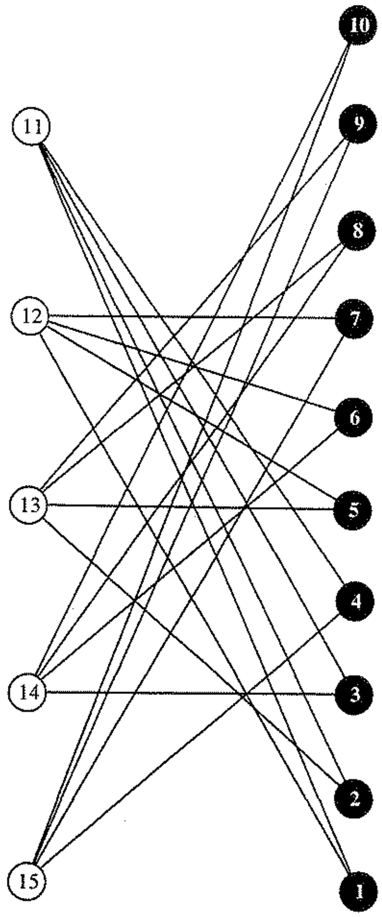
So, let Φ take the black vertices of P(N,1,b) to the black vertices of Distance3(P(N,1,N-b)) in the following way: a black vertex b_1 of P(N,1,b) goes to a black vertex b_2 of Distance3(P(N,1,N-b)) if the set to which b_2 corresponds contains exactly those elements of {1, ..., N} which are NOT elements of the set to which b_1 corresponds (in other words, the complement set.) Since, in P(N,1,b), b_1 corresponds to a set of length b, then there is exactly one complementary set of length N-b in Distance3(P(N,1,N-b)). This means that Φ is one-to-one and onto.

Now suppose that $b_1 w_1$ is an edge of P(N,1,b), then the set to which w_1 corresponds is a subset of the set to which b_1 corresponds. By definition of Φ, it follows that $\Phi(w_1) = w_1'$ where the singleton set to which w_1' corresponds is the same singleton set to which w_1 corresponds. Also by definition of , it follows that $\Phi(b_1) = b_1'$ where the set to which b_1' corresponds contains exactly those elements not in the set to which b_1 corresponds. It follows that $\Phi(b_1)\Phi(w_1) = b_1' w_1'$ is an edge of Distance3(P(N,1,N-b)). Therefore Φ preserves adjacency, and it follows that Φ is an isomorphism. é

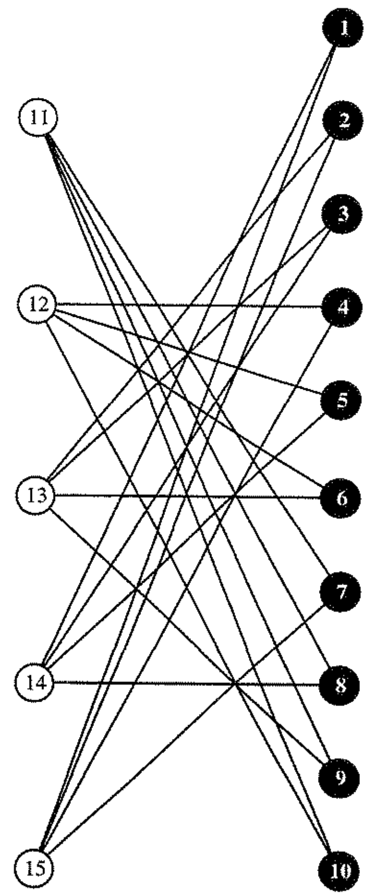
Here is an example to illustrate this isomorphism:

Isomorphism between P(5,1,2) and Distance3(P(5,1,3)):

SINGLETON SETS	SETS OF LENGTH 2	
11 → 11	1 → 10	6 → 5
12 → 12	2 → 9	7 → 4
13 → 13	3 → 8	8 → 3
14 → 14	4 → 7	9 → 2
15 → 15	5 → 6	10 → 1



P(5,1,2)



Distance3(P(5,1,3))

It is true that the wedge product of a power set graph with a = 1 and a complete bipartite graph has similar properties. For instance, notice from the last column of the table in Appendix E that the distance between a (B,B) ((W,W)) vertex and a (W,W) ((B,B)) vertex is less than or equal to three. (In fact, it is either one or three.) Also notice from Appendix D the similarities between the graph $P(N,1,b) \wedge K_{x,y}$ and the graph Distance3($P(N,1,N\text{-}b) \wedge K_{x,y}$). These observations lead to the next lemma and theorem.

THEOREM: In the graph $P(N,1,b) \wedge K_{x,y}$, where $bx = C(N-1,b-1)y$, a (W,W) ((B,B)) vertex will be at a distance 1 or 3 from any of the (B,B) ((W,W)) vertices.

Proof:

Consider Γ (the (B,B) and (W,W) vertices of $P(N,1,b) \wedge K_{x,y}$ and the edges joining them.) Now because Γ is a bipartite graph, then a (B,B) ((W,W)) vertex is at an odd distance from all of the (W,W) ((B,B)) vertices.

Without loss of generality, suppose that in P(N,1,b), the white vertices are those corresponding to the singleton sets, and in $K_{x,y}$ suppose that the white vertices are those *y* vertices of degree *x*. It follows that there are *Ny* (W,W) vertices. Because $\Gamma' \wedge K_{x,y}$, where Γ' is a bi-transitive graph, is bi-transitive, it follows clearly that Γ is bi-transitive. Furthermore, Γ is transitive among the (B,B) vertices and similarly the white vertices. So, without loss of generality, consider the (W,W) vertex corresponding to the ordered pair $(\{1\}, w_1')$. By definition of Γ, the vertex corresponding to the ordered pair $(\{1\}, w_1')$ will be adjacent to (at a distance 1 from) the *C(N-1,b-1)x* (B,B) vertices corresponding to the ordered pairs (b_i,b_j') where $1 \le i \le C(N-1,b-1)$ and $1 \le j \le x$ and where each b_i is a vertex in P(N,1,b) which corresponds to a set of length *b* containing the element *1*. Among these sets of length *b* containing the element *1* are the following sets: {1, ..., b-1, b}, {1, ..., b-1, b+1}, ..., {1, ..., b-1, N}. Since, among these sets are all other elements of the set {1, ..., N} besides the element 1, then the vertices corresponding to these sets collectively are adjacent to the vertices corresponding to all the other singleton sets {2}, ..., {N} in P(N,1,b). Since any b_j' for $1 \le j \le x$ is adjacent to all of the white vertices in $K_{x,y}$, it follows that the vertices in Γ corresponding to the ordered pairs $(\{1, ..., b-1, b\},b_1')$, $(\{1, ..., b-1, b+1\},b_1')$, ..., $(\{1, ..., b-1, N\},b_1')$ are adjacent tot he verices correpsonding to the ordered pairs $(\{2\},w_n')$, ..., $(\{N\},w_n')$ where $1 \le n \le y$. These vertices will be at a distance 2 from the vertex corresponding to the ordered pair $(\{1\}, w_1')$.

Now consider the $C(N-1,b)x$ (B,B) vertices to which the vertex corresponding to the ordered pair ({1},w_1') is not adjacent. These vertices correspond to the following ordered pairs: ({2, ..., b+1},w_i'), ({2, ..., b, b+2},w_i'), ..., ({2, ..., b,N},w_i'), ({2, 4, ...,b+2},w_i'), ..., ({2, 4, ..., b+1, N},w_i'), ..., ({2, (N-b)+2, ..., N},w_i'), ..., ({(N-b)+1, ..., N},w_i') where $1 \leq i \leq x$. Now because all of the vertices corresponding to these ordered pairs are adjacent to at least one of the vertices corresponding to the oredered pairs ({2},w_n'), ..., ({N},w_n') where $1 \leq n \leq y$, it follows that the vertices corresponding to these ordered pairs are at a distance 3 from the vertex corresponding to the ordered pair ({1}, w_1').

Because $C(N-1,b)x + C(N-1,b-1)x = C(N,b)x$, it follows that the vertex corresponding to the ordered pair ({1}, w_1') is at a distance one or three from the $C(N,b)x$ (B,B) vertices. It furthermore follows that any (W,W) vertex is at a distance one or three from any of the (B,B) vertices. Because $d(u,v) = d(v,u)$, it follows that any (B,B) vertex is at a distance one or three from any of the (W,W) vertices. é

THEOREM: $P(N,1,b) \wedge K_{x,y} \cong$ Distance3($P(N,1,N-b) \wedge K_{x,y}$)

Proof:

As in the previous proof, in any wedge product $P(N,1,b) \wedge K_{x,y}$, a (B,B) ((W,W)) vertex is at a distance 1 or 3 from any other the (W,W) ((B,B)) vertices. Furthermore, using the convention that the (B,B) vertices are those corresponding to the ordered pairs (b_i,b_j') where b_i is a vertex in P(N,1,b) corresponding to a set of length *b*, and b_j' is one of the *x* black vertices of of degree *y*, and where $1 \leq i \leq C(n,b)$ and $1 \leq j \leq x$. It follows that the (W,W) vertices are those corresponding to the ordered pairs (w_n, w_m') where $1 \leq n \leq N$ and $1 \leq m \leq y$, and where w_n is a vertex of P(N,1,b) corresponding to a singleton set, and w_m' is one of the *y* white vertices of $K_{x,y}$ of degree *x*. It follows that the (B,B) vertices of $P(N,1,b) \wedge K_{x,y}$ are of degree *by* and the (W,W) vertices are of degree *C(N-1,b-1)x*.

Consider the graph $P(N,1,b) \wedge K_{x,y}$. The (B,B) vertices are of degree

(N-b)y, and the (W,W) vertices are of degree *C(N-1,(N-b)-1)x.* Since (B,B) ((W,W)) vertices are at a distance 1 or 3 from any of the (W,W) ((B,B)) vertices, in the graph Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$), the (B,B) ((W,W)) vertices will be adjacent to those (W,W) ((B,B)) vertices to which they are not adjacent to in P(N,1,N-b) $\wedge$ $K_{x,y}$. So the (B,B) vertices of Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$) will have degree *Ny-(N-b)y = by* and the (W,W) vertices will have degree *C(N,N-b)x - C(N-1,(N-b)-1)x = C(N-1,N-b)x = C(N-1,b-1)x.* So in both graphs P(N,1,b) $\wedge$ $K_{x,y}$ and Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$), there are *C(N,b)x = C(N,N-b)x* (B,B) vertices of degree *by* and there are *Ny* (W,W) vertices of degree *C(N-1,b-1)x.*

Let Φ: V(P(N,1,b) $\wedge$ $K_{x,y}$) $\rightarrow$ V(Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$)) take the (W,W) vertices of P(N,1,b) $\wedge$ $K_{x,y}$ to the (W,W) of Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$) in the following way. If v({a},r_i) means the vertex corresponding ordered pair in which {a} is a singelton set in P(N,1,b) and r_i is a white vertex of $K_{x,y}$ where $1 \leq i \leq y$, then under Φ : v({1},r_i) $\rightarrow$ v({1},r_i), ..., v({N},r_i) $\rightarrow$ v({N},r_i) for $1 \leq i \leq y$.

Now, in P(N,1,b) $\wedge$ $K_{x,y}$, a (W,W) vertex is adjacent to a (B,B) vertex if the singleton set of the ordered pair to which the (W,W) vertex corresponds is a subset of the set of length *b* to which the (B,B) vertex corresponds. In Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$) a (W,W) vertex is adjacent to a (B,B) vertex if the singleton set of the ordered pair to which the (W,W) vertex corresponds is NOT a subset of the set of length (N-b) of the ordered pair to which the (B,B) vertex corresponds.

So, let Φ take the (B,B) vertices of P(N,1,b) $\wedge$ $K_{x,y}$ to the (B,B) vertices of Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$) in the following way: Suppose (b_s, b_t') is a (B,B) vertex of P(N,1,b) $\wedge$ $K_{x,y}$ where $1 \leq s \leq$ C(N,b) and $1 \leq t \leq x$. Suppose (b_u,b_v') is a (B,B) vertex of Distance3(P(N,1,N-b) $\wedge$ $K_{x,y}$) where $1 \leq u \leq$ C(N, N-b) = C(N,b) and $1 \leq v \leq x$. Then Φ(b_s, b_t') = (b_u,b_v') if the set tho which b_u corresponds contains exactly those elements of {1, ..., N} which are NOT elements of the set to which b_s corresponds (in other words the complementary set). Since, in P(N,1,b) $\wedge$ $K_{x,y}$, b_s of

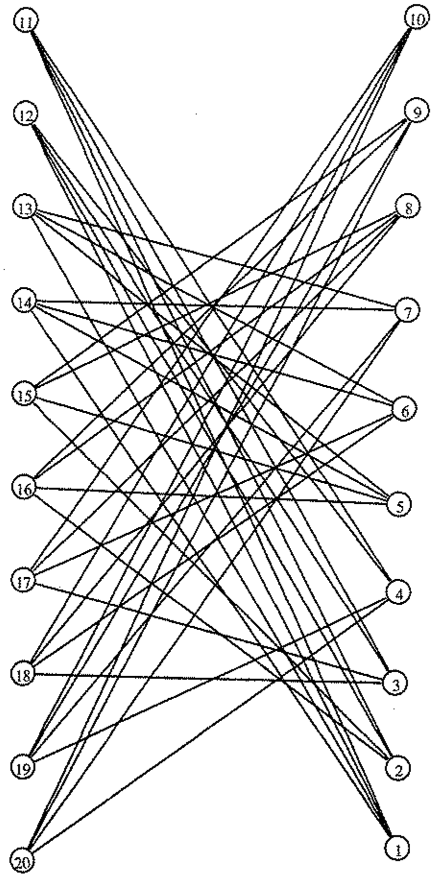
(b_s, b_t') corresponds to a set of length b, then there is exactly one complementary set of length N-b. It follows that there is exactly one vertex (b_u, b_v') which is adjacent to the vertex which corresponds to (b_s, b_t'). This means that Φ is one-to-one and onto.

Now suppose that $b_1 = (b_{1_1}, b_{1_1}')$, $w_1 = (w_{1_1}, w_{1_1}')$ and $b_1 w_1$ is an edge of $P(N,1,b) \wedge K_{x,y}$, then the set to which w_{1_1} corresponds is a subset of the set to which b_{1_1} corresponds and b_{1_1}' and w_{1_1}' are any of the black and white vertices of $K_{x,y}$ respectively. By definition of Φ, it follows that $\Phi(w_1) = w_2 = (w_{2_1}, w_{2_1}')$ where $w_{1_1} = w_{2_1}$ and $w_{1_1}' = w_{2_1}'$. Also by definition of Φ, it follows that $\Phi(b_1) = b_2 = (b_{2_1}, b_{2_1}')$ where $b_{1_1}' = b_{2_1}'$.and where the set to which b_{2_1} corresponds contains exactly those elements not in the set to which b_{1_1} corresponds. Now in Distance3$(P(N,1,N\text{-}b) \wedge K_{x,y})$. a (B,B) vertex $b_3 = (b_{3_1}, b_{3_1}')$ is adjacent to a (W,W) vertex $w_3 = (w_{3_1}, w_{3_1}')$ if the singleton set to which w_{3_1} is not a subset of the set to which b_{3_1} corresponds, otherwise it would be adjacent and at a distance 1. It follows that $\Phi(b_1)\Phi(w_1) = b_2 w_2$ is an edge of Distance3$(P(N,1,N\text{-}b) \wedge K_{x,y})$. Therefore Φ preserves adjacency, and it follows that Φ is an isomorphism. é

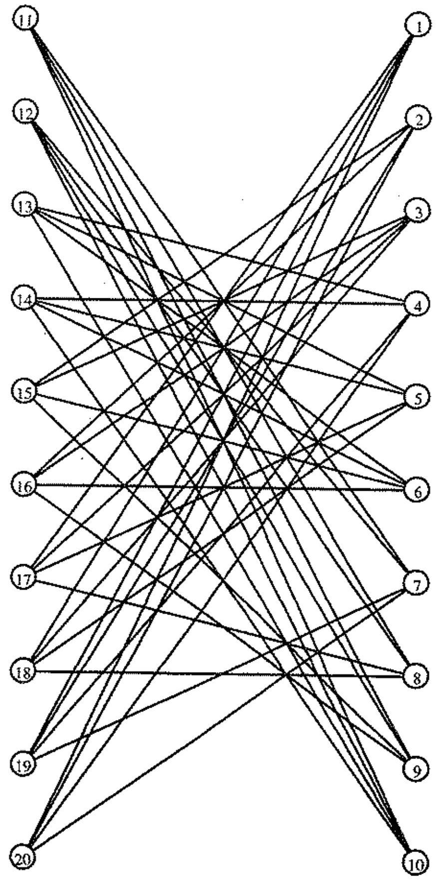
Here is an example which illustrates this isomorphism:

$P(5,1,2) \wedge K_{1,2}$ and Distance3$(P(5,1,3) \wedge K_{1,2})$

Singleton sets		Sets of length 2	
11 → 11	16 → 16	1 → 10	6 → 5
12 → 12	17 → 17	2 → 9	7 → 4
13 → 13	18 → 18	3 → 8	8 → 3
14 → 14	19 → 19	4 → 7	9 → 2
15 → 15	20 → 20	5 → 6	10 → 1



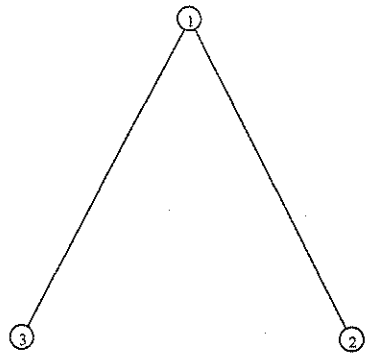
$P(5,1,2) \wedge K_{1,2}$



$\text{Distance3}(P(5,1,3) \wedge K_{1,2})$

Carets

Definition: A *caret*, denoted *312*, is an object of a graph in which the two edges (3,1) and (1,2) meet at vertex 1.



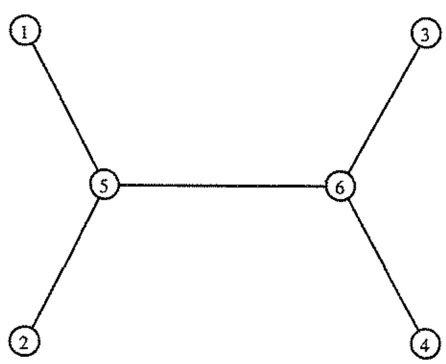
Caret 312

Definition: Let Γ_4 be the graph whose vertices are the carets and darts of the graph Γ. The edges of Γ_4 will join a dart to a caret if the dart is part of the caret. (Hence, there are 4 darts to every caret.)

LEMMA: In Γ_4 if 56 and 65 are darts, then there is some symmetry $\alpha \in G(\Gamma_4)$ such that $(56)\alpha = 65$ and $(65)\alpha = 56$ (α switches the darts).

Proof:

Suppose that in Γ, vertex 5 is adjacent to the vertices 1,2, and 6, and suppose that vertex 6 is adjacent tothe vertices 3, 4, and 5:



In Γ

By definition of Γ_4, we know that a dart is adjacent to the caret if the dart is on an edge of the the caret in Γ. So it follows that in Γ_4, the dart 56 is adjacent to the carets 156, 256, 365, and 465. SImilarly, we have that the dart 65 is adjacent tot he carets 156, 256, 365, and 465.

Suppose that α switches 56 and 65 while leaving all other vertices in Γ_4 fixed. It must be shown that α is a symmetry of Γ_4. There are two different kinds of edges: those which are incident to either 65 or 56, and those which are not. Any edge which is not incident to either 65 or 56 is fixed by α $((736,36)\alpha = (736,36))$. Consider the other edges which are incident to either 56 or 65. Without loss of generality, suppose that (156,56) is one of these edges. Then $(156,56)\alpha = (156,65) \in \Gamma_4$. So α takes any edge of Γ_4 to another edge of Γ_4. Therefore $\alpha \in G(\Gamma_4)$. é

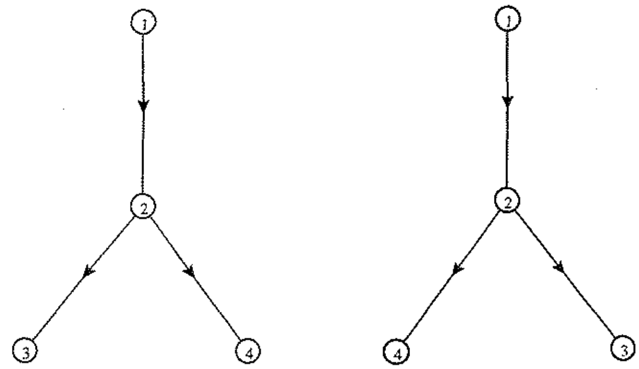
Definition: An *n-path*, denoted $v_0v_1 \ldots v_n$, is a alternating sequence of vertices and edges in which the edge is incident with the two vertices immediately preceding and following it.

Definition: A graph Γ is said to be *n-path transitive*, $n \geq 1$, if it has an n-path such that for any two n-paths, there exists an automorphism of $G(\Gamma)$ sending one n-path to the other. The next lemma illustrates an equivalence of a graph being 2-path transitive.

LEMMA: Suppose that Γ is a 3-valent dart transitive graph, and suppose that at any vertex 2, there is some $\sigma \in G(\Gamma)$ such that the dart 12 is fixed under σ, and $(23)\sigma = 24$. Then Γ is 2-path transitive.

Proof:

Suppose that 123 and 456 are 2-paths of Γ. Since Γ is vertex-transitive, then there is some $\tau \in G(\Gamma)$ such that $2\tau = 5$. Given that 1, 3, and x are the vertices adjacent to 2 and that 4, 6, and y are the vertices adjacent to 5, then because Γ is dart-transitive, there is some symmetry $\tau' \in G(\Gamma)$, such that $(12)\tau' = 45$. Under this τ' two things could occur: either the dart 23 could go to the dart 5y or the dart 23 could go to the



Action of σ

dart 56. In the latter case, we are done: τ' takes the 2-path 123 to the 2-path 456. In the former case, the 2-path 123 goes to the 2-path 45y. But if the symmetry $\sigma \in G(\Gamma)$ is first applied at the vertex 2, fixing the dart 12, then the darts 23 and 2x could be switched. So the composition of $\sigma\tau' \in G(\Gamma)$ takes the 2-path 123 to the 2-path 456. Therefore, there dies exist a symmetry in $G(\Gamma)$ that takes the 2-path 123 to the 2-path 456. Hence Γ is 2-path transitive.

LEMMA: Let Γ be a 3-valent dart-transitive graph. Then Γ is caret-transitive.

Proof:

Suppose that abc and def are carets of Γ. Suppose now that a,c, and x are the vertices adjacent to b and suppose that d,f,and y are the vertices adjacent to e. Because Γ is vertex-transitive, there is some $\sigma \in G(\Gamma)$ such that $b\sigma = e$. Given this, it is true that the edge (a,b) can be sent to the edges (d,e), (f,e), and (y,e) because Γ is edge-transitive. Let us consider each case:

<u>Case 1</u>:

Suppose that under $\sigma_1 \in G(\Gamma)$, $b\sigma_1 = e$ as with σ and $a\sigma_1 = d$, then either c can be sent to f or y. If σ_1: $c \to f$, then we are done. $(abc)\sigma_1 =$ (def) and Γ is caret preserving. Otherwise, σ_1: $c \to y$. Let us call this permutation σ_1': $a \to d$, $b \to e$, $c \to y$, and $x \to f$ and we will consider this case later.

Case 2:

Suppose that under $\sigma_2 \in G(\Gamma)$, $b\sigma_2 = e$ as with σ and $a\sigma_2 = f$, then either c can be sent to d or y. If σ_2: $c \to d$, then we are done. $(abc)\sigma_2 = (fed) = (def)$ and Γ is caret preserving. Otherwise, σ_2: $c \to y$. Let us call this permutation σ_2': $a \to f$, $b \to e$, $c \to y$, and $x \to d$ and we will consider this case later.

Case 3:

Suppose that under $\sigma_3 \in G(\Gamma)$, $b\sigma_3 = e$ as with σ and $a\sigma_3 = y$, then either c can be sent to d or f. Neither case will give us that Γ is caret transitive. But, let us define these permutations for further use:

σ_3: $a \to y$, $b \to e$, $c \to d$, and $x \to f$ and

σ_3': $a \to y$, $b \to e$, $c \to f$, and $x \to d$.

At least 3 of the above permutations must be in the symmetry group of Γ. If σ_1 or $\sigma_2 \in G(\Gamma)$, then we have already discovered that Γ is caret transitive. Suppose that neither is in $G(\Gamma)$. Then σ_1' and $\sigma_2' \in G(\Gamma)$ and at least one of σ_3 and $\sigma_3' \in G(\Gamma)$. Suppose that $\sigma_3 \in G(\Gamma)$, then it follows that $\sigma_1' \sigma_2'^{-1} \sigma_3 \in G(\Gamma)$ which would mean that Γ is caret transitive. Now suppose that $\sigma_3' \in G(\Gamma)$, then it follows that $\sigma_1' \sigma_2'^{-1} \sigma_3' \in G(\Gamma)$ which would mean that Γ is caret transitive. Since we have exhausted all of the cases it can be concluded that Γ is caret transitive.

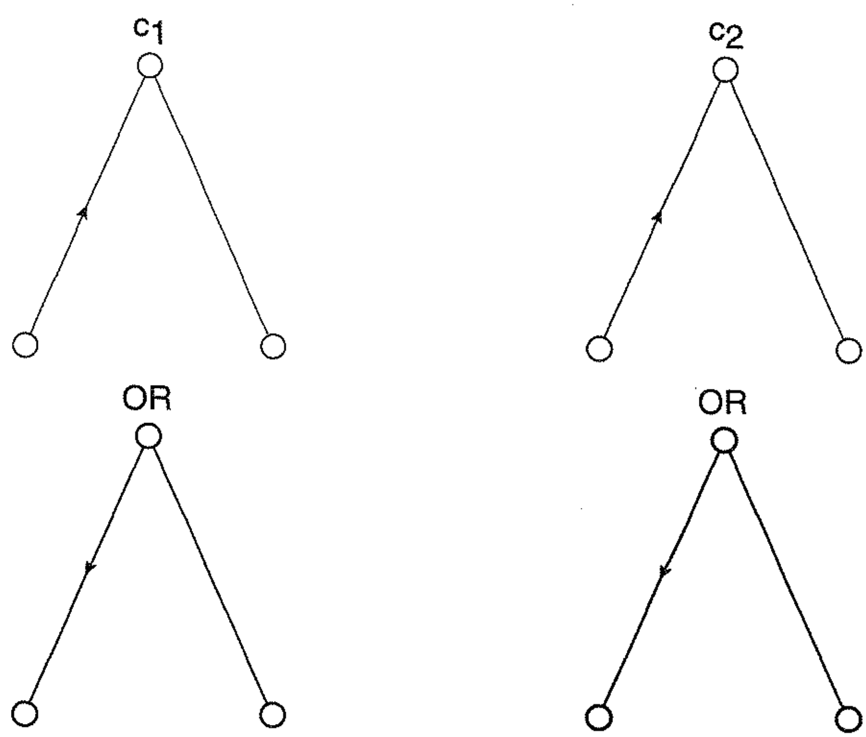
LEMMA: Let Γ be a 3-valent 2-path transitive graph. Then Γ_4 of Γ will be edge-transitive.

Proof:

Define σ on Γ_4 by $(c,d)\sigma = (c\sigma,d\sigma)$ where dart d is a part of caret c. Suppose that $(c_1,d_1) \in G(\Gamma_4)$ and let $c_1 = (x_1,y_1,z_1)$. It follow that d1 can be any of the darts x_1y_1, y_1x_1, y_1z_1, or z_1y_1. Because Γ is caret transitive, we know that $c_1\sigma = c_2 = (x_2,y_2,z_2)$ is a caret of Γ and from the definition of σ we have that $x_1\sigma = x_2$, $y_1\sigma = y_2$, $z_1\sigma = z_2$. Now $d_1\sigma = d_2$ could be any of the darts: $(x_1y_1)\sigma = x_2y_2$, $(y_1x_1)\sigma = y_2x_2$, $(y_1z_1)\sigma = y_2z_2$, $(z_1y_1)\sigma = z_2y_2$. Because all of these darts are part of c_2, we have that they are all connected to c_2 in Γ_4. Therefore $(c_2,d_2) = (c_1\sigma, d_1\sigma) = (c_1,d_1)\sigma$. So σ is a symmetry of Γ_4.

Suppose that $e_1 = (c_1,d_1)$ and $e_2 = (c_2,d_2)$ where c_1 and c_2 are carets of and d_1 and d_2 are darts which are part of an edge of carets c_1 and c_2 respectively. Then d_1 can either be an incoming dart (pointing towards the tip of the caret) or an outgoing (pointing away from the tip of the caret). Similarly, d_2 can either be an incoming dart or an outgoing dart. Because, given any caret xyz, this caret can be viewed as zyx in Γ, we can, without loss of generality, consider as the incoming dart xy or zy and similarly we can without loss of generality consider as the outgoing dart yx or yz.

Suppose that there is some $\tau \in G(\Gamma)$ such that $c_1\tau = c_2$, then by the reasoning above, four separate cases must be considered for d_1 and d_2 (d_1 is the directed dart in the left graphs and d_2 is the directed dart in the right graphs):



Suppose that corresponding to τ in $G(\Gamma)$, there is some τ' in $G(\Gamma_4)$

such that $c_1\tau' = c_2$.

Case1: d_1 and d_2 are both incoming darts of carets c_1 and c_2 respectively. If under τ, d_1 goes to d_2, then we are done. Suppose now that τ takes d_1 to the other incoming dart of c_2. Because Γ is 2-path transitive, there is some $\alpha \in G(\Gamma)$ which switches the incoming darts of c_2. So there is some $\alpha' \in G(\Gamma_4)$ which switches the incoming darts of c_2. Hence, $d_1\tau'\alpha' = d_2$ and $\sigma = \tau'\alpha' \in G(\Gamma_4)$.

Case2: d_1 is an incoming dart of c_1 and d_2 is an outgoing dart of caret c_2. Suppose under τ, d_1 goes tothe edge containing d_2. Because there is $\beta' \in G(\Gamma_4)$ which switches the two darts on the same edge. So $d_1\tau'\beta' = d_2$ and $\sigma = \tau'\beta' \in G(\Gamma_4)$. Now consider that under τ, d_1 goes to the other edge of caret c_2. Because Γ is 2-path transitive, there is some $\gamma \in G(\Gamma)$ and $\gamma' \in G(\Gamma_4)$ which switch d_1 with the other incoming dart. There is also some $\delta' \in G(\Gamma_4)$ which switches the two darts d_2 and $\gamma'(d_1)$. So it follows that $d_1\tau'\gamma'\delta' = d_2$ and $\sigma = \tau'\gamma'\delta' \in G(\Gamma_4)$.

Case3: d_1 and d_2 are both outgoing darts of carets c_1 and c_2 respectively. If under τ, d_1 goes to d_2, then we are done. Suppose now that τ takes d_1 to the other outgoing dart of c_2. Because Γ is 2-path transitive, there is some $\varepsilon \in G(\Gamma)$ which switches the outgoing darts of c_2. So there is some $\varepsilon' \in G(\Gamma_4)$ which switches the incoming darts of c_2. Hence, $d_1\tau'\varepsilon' = d_2$ and $\sigma = \tau'\varepsilon' \in G(\Gamma_4)$.

Case4: d_1 is an outgoing dart of c_1 and d_2 is an incoming dart of caret c_2. Suppose under τ, d_1 goes to the edge containing d_2. Because there is $\lambda' \in G(\Gamma_4)$ which switches the two darts on the same edge, $d_1\tau'\lambda' = d_2$ and $\sigma = \tau'\lambda' \in G(\Gamma_4)$. Now consider that under τ, d_1 goes to the other edge of caret c_2. Because Γ is 2-path transitive, there is some $\mu \in G(\Gamma)$ and $\mu' \in G(\Gamma_4)$ which switch d_1 with the other outgoing dart. There is also some $\nu' \in G(\Gamma_4)$ which switches the two darts d_2 and $\mu'(d_1)$. So it follows that $d_1\tau'\mu'\nu' = d_2$ and $\sigma = \tau'\mu'\nu' \in G(\Gamma_4)$.

Since we have exhausted all cases, we have shown that there is some $\sigma \in G(\Gamma_4)$ which takes e_1 to e_2. Therefore, Γ_4 is edge-transitive.

THEOREM: Let Γ be a 3-valent 2-path-transitive graph. Then Γ_4 of Γ will be a 4-valent strictly bi-transitive graph.

Proof:

It is true that Γ_4 of any graph will be 4-valent because each caret contains 4 darts, and each dart is contained in 4 carets (Each vertex has degree 3 which means that each vertex is the center of 3 carets which means that two of these carets contain the same particular edge.) Since the edge contains 2 vertices, we end up with 4 carets to every dart.

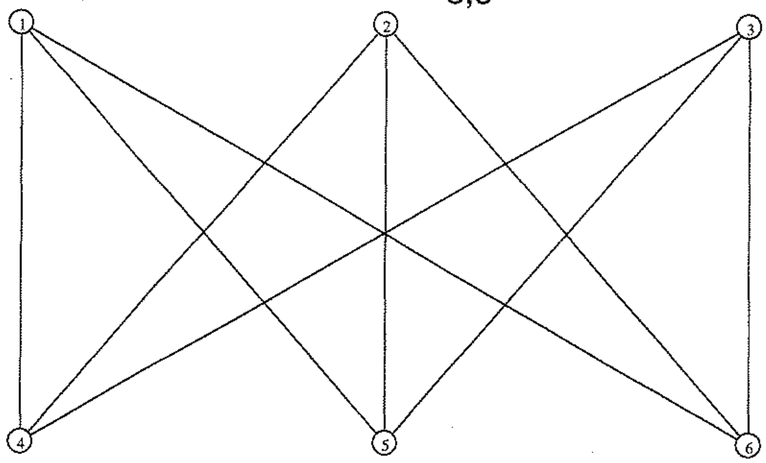
By definition of Γ_4, it is true that an edge joins a caret to a dart. Therefore, there is no such edge joining any two darts and there is no such edge joining any two carets. Therefore if $V_1 = \{$ all carets of $\Gamma_4\}$ and $V_2 = \{$ all darts of $\Gamma_4\}$, then Γ_4 is bipartite and V_1 and V_2 are the partitions of vertices of Γ_4.

From the previous lemma, it is true that Γ_4 is edge-transitive. So now all that needs to be shown is that Γ_4 is not vertex transitive. So suppose that Γ_4 is vertex-transitive. Then it is true that there is some $\gamma \in G(\Gamma_4)$ such that $(abc)\gamma = (b,a)$ where (abc) is a caret and (b,a) is a dart which is part of an edge of (abc).

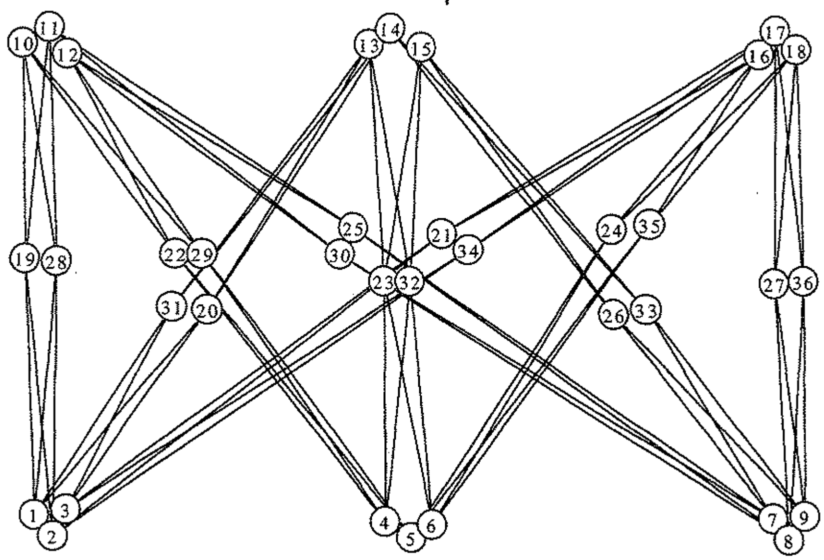
Since γ must take an edge of Γ_4 to an edge of Γ_4, we have that dart $(a,b)\gamma$ could be one of the carets: (abc), (bac), (abd), or (bad). SImilarly, $(b,a)\gamma$ could also be one of the carets: (abc), (bac), (abd), or (bad). So $(a,b)\gamma$ and $(b,a)\gamma$ must be connected to the same four darts. This is only possible if $(a,b)\gamma = (b,a)\gamma$ because no two of (abc), (bac), (abd), or (bad) are connected to the same four darts. Therefore, γ is not an automorphism because it is not one-to-one. It follows that there is no symmetry in $G(\Gamma_4)$ that takes carets to darts and darts to carets. Therefore Γ_4 is not vertex-transitive and Γ_4 is strictly bi-transitive by definition. é

Here is an example to illustrate this Γ_4 graph.

$\Gamma = K_{3,3}$



Γ_4



Appendix A

Power Set Graphs

Graph	N	a	b	# of top vertices	Degree of top vertices	# of bottom vertices	Degree of bot. vertices	Order of the group	Distance
P(3,1,2)	**3**	**1**	**2**	**3**	**2**	**3**	**2**	**12**	**≤3**
P(4,1,2)	4	1	2	6	2	4	3	24	≤3
P(4,1,3)	**4**	**1**	**3**	**4**	**3**	**4**	**3**	**48**	**≤3**
P(5,1,2)	5	1	2	10	2	5	4	120	≤3
P(5,1,3)	5	1	3	10	3	5	6	120	≤3
P(5,1,4)	**5**	**1**	**4**	**5**	**4**	**5**	**4**	**240**	**≤3**
P(5,2,3)	**5**	**2**	**3**	**10**	**3**	**10**	**3**	**240**	**≤5**
P(6,1,2)	6	1	2	15	2	6	5	720	≤3
P(6,1,3)	6	1	3	20	3	6	10	720	≤3
P(6,1,4)	6	1	4	15	4	6	10	720	≤3
P(6,1,5)	**6**	**1**	**5**	**6**	**5**	**6**	**5**	**1440**	**≤3**
P(6,2,3)	6	2	3	20	3	15	4	720	≤5
P(6,2,4)	**6**	**2**	**4**	**15**	**6**	**15**	**6**	**1440**	**≤3**
P(7,1,2)	7	1	2	21	2	7	6	5040	≤3
P(7,1,3)	7	1	3	35	3	7	15	5040	≤3
P(7,1,4)	7	1	4	35	4	7	20	5040	≤3
P(7,1,5)	7	1	5	21	5	7	15	5040	≤3
P(7,1,6)	**7**	**1**	**6**	**7**	**6**	**7**	**6**	**10080**	**≤3**
P(7,2,3)	7	2	3	35	3	21	5	5040	≤5
P(7,2,4)	7	2	4	35	6	21	10	5040	≤3
P(7,2,5)	**7**	**2**	**5**	**21**	**10**	**21**	**10**	**10080**	**≤3**
P(7,3,4)	**7**	**3**	**4**	**35**	**4**	**35**	**4**	**10080**	**≤7**
P(8,1,2)	8	1	2	28	2	8	7	40320	≤3
P(8,1,3)	8	1	3	56	3	8	21	40320	≤3
P(8,1,4)	8	1	4	70	4	8	35	40320	≤3
P(8,1,5)	8	1	5	56	5	8	35	40320	≤3
P(8,1,6)	8	1	6	28	6	8	21	40320	≤3
P(8,1,7)	**8**	**1**	**7**	**8**	**7**	**8**	**7**	**80640**	**≤3**
P(8,2,3)	8	2	3	56	3	28	6	40320	≤5
P(8,2,4)	8	2	4	70	6	28	15	40320	≤3
P(8,2,5)	8	2	5	56	10	28	20	40320	≤3
P(8,2,6)	**8**	**2**	**6**	**28**	**15**	**28**	**15**	**80640**	**≤3**

P(8,3,4)	8	3	4	70	4	56	5	40320	≤7
P(8,3,5)	**8**	**3**	**5**	**56**	**10**	**56**	**10**	**80640**	**≤5**

Graph	N	a	b	# of top vertices	Degree of top vertices	# of bottom vertices	Degree of bot. vertices	Order of the group	Distance
P(9,1,2)	9	1	2	36	2	9	8	362880	≤3
P(9,1,3)	9	1	3	84	3	9	28	362880	≤3
P(9,1,4)	9	1	4	126	4	9	56	362880	≤3
P(9,1,5)	9	1	5	126	5	9	70	362880	≤3
P(9,1,6)	9	1	6	84	6	9	56	362880	≤3
P(9,1,7)	9	1	7	36	7	9	28	362880	≤3
P(9,1,8)	**9**	**1**	**8**	**9**	**8**	**9**	**8**	**725760**	**≤3**
P(9,2,3)	9	2	3	84	3	36	7	362880	≤5
P(9,2,4)	9	2	4	126	6	36	21	362880	≤3
P(9,2,5)	9	2	5	126	10	36	35	362880	≤3
P(9,2,6)	9	2	6	84	15	36	35	362880	≤3
P(9,2,7)	**9**	**2**	**7**	**36**	**21**	**36**	**21**	**725760**	
P(9,3,4)	9	3	4	126	4	84	6		
P(9,3,5)	9	3	5	126	10	84	15		
P(9,3,6)	**9**	**3**	**6**	**84**	**20**	**84**	**20**		
P(9,4,5)	**9**	**4**	**5**	**126**	**5**	**126**	**5**		
P(10,1,2)	10	1	2	45	2	10	9		
P(10,1,3)	10	1	3	120	3	10	36		
P(10,1,4)	10	1	4	210	4	10	84		
P(10,1,5)	10	1	5	252	5	10	126		
P(10,1,6)	10	1	6	210	6	10	126		
P(10,1,7)	10	1	7	120	7	10	84		
P(10,1,8)	10	1	8	45	8	10	36		
P(10,1,9)	**10**	**1**	**9**	**10**	**9**	**10**	**9**		
P(10,2,3)	10	2	3	120	3	45	8		
P(10,2,4)	10	2	4	210	6	45	28		
P(10,2,5)	10	2	5	252	10	45	56		
P(10,2,6)	10	2	6	210	15	45	70		
P(10,2,7)	10	2	7	120	21	45	56		
P(10,2,8)	**10**	**2**	**8**	**45**	**28**	**45**	**28**		
P(10,3,4)	10	3	4	210	4	120	7		
P(10,3,5)	10	3	5	252	10	120	21		
P(10,3,6)	10	3	6	210	20	120	35		
P(10,3,7)	**10**	**3**	**7**	**120**	**35**	**120**	**35**		

Appendix B

```
(**************Define   the   complete   bipartite*************)
        Clear[u,v,q,r,f,g,top,bottom,edgelist,totalvert,newlist]
        completeBipartite[u_,v_]:=
           (
           top=Range[u];
           bottom=Range[v];
           totalvert=Join[top,  bottom];
           edgelist=Table[List[g],{g,1,(u+v)}];

           For[f=1,  f<=Length[edgelist],  f++,
              If[f<=u,
              For[q=(u+1),q<=Length[edgelist],q++,
                     AppendTo[edgelist[[q]],f]
                ],
              For[r=1,r<=u,r++,
                     AppendTo[edgelist[[r]],f]
               ]
              ]
            ];

           edgelist
           )

        (**************Define   the   power   set
graph****************)
        Clear[n,a,b,set,perm,takeA,list,aperm,sort,sortperma,
        perma, takeB, bperm,sortpermb, permb, totalvertices,
         h,k,x,y,l,edges];

        powergraph[n_,a_,b_]:=
        (

        BeginPackage["DiscreteMath`Combinatorica`"];
        KSubsets[l_List,0] := { {} };
        KSubsets[l_List,1]  :=  Partition[l,1];
        KSubsets[l_List,k_Integer?Positive]  :=  {l}  /;  (k  ==  Length[l]);
        KSubsets[l_List,k_Integer?Positive]  :=  {}   /;  (k  >  Length[l]);
```

```
KSubsets[l_List,k_Integer?Positive]  :=
   Join[
      Map[(Prepend[#,First[l]])&,  KSubsets[Rest[l],k-1]],
      KSubsets[Rest[l],k]
   ];



set:=Range[n];
perma=KSubsets[set,a];
permb=KSubsets[set,b];

totalvertices=Join[permb,perma];

edges=Table[List[h],{h,1,Length[totalvertices]}];

la=Length[perma];
lb=Length[permb];
lt=Length[totalvertices];


For[x=1,  x<=lb,x++,
   For[y=(1+lb),  y<=lt,y++,

      If[SameQ[Union[Join[totalvertices[[x]],
                                                   totalvertices[[y]]
                                              ]
                                    ],  totalvertices[[x]]
                           ],
                             AppendTo[edges[[x]],y];

                             AppendTo[edges[[y]],x];

EndPackage[ ];

edges
```

```
(*******************   Define   wedge   *****************)
Clear[newlist,newlistB,newlistW,wedgelist,position,i,ii,j,jj,
         s,ss,t,tt,c,cc,d,dd,e,ee,z,zz,lengthBB,constant,lbottom,
          lnewlist,llist1,llist2,b1,b2,list1,list2,wedge]

wedge[list1_,list2_,b1_,b2_]:=
       (
       llist1=Length[list1];
       llist2=Length[list2];
       lbottom=llist2-b2;
       lengthBB=b1*b2;

       newlistB=Flatten[Table[List[list1[[i]],list2[[j]]],
            {i,1,b1},{j,1,b2}],1];


       newlistW=Flatten[Table[List[list1[[ii]],list2[[jj]]],
            {ii,b1+1,llist1},
            {jj,b2+1,llist2}],1];


       newlist=Join[newlistB,newlistW];

       wedgelist=Table[List[z],
            {z,1,Length[newlistB]+Length[newlistW]}];

       lnewlist=Length[newlist];
       constant=-1+lengthBB-b1*lbottom-lbottom;

       For[c=1, c<=lnewlist, c++,
          For[d=2, d<=Length[newlist[[c,1]]], d++,
             For[e=2, e<=Length[newlist[[c,2]]], e++,
                  If[newlist[[c,1,d]]<=b1,
                         position=(newlist[[c,1,d]]-1)*b2+e-1,
                         position=constant+e+newlist[[c,1,d]]*lbottom
                   ];

                  AppendTo[wedgelist[[c]],position];
             ]
          ]
```

```
   ];

wedgelist

wedge[list1_,list2_]:=
(
llist1=Length[list1];
llist2=Length[list2];

newlist=Flatten[Table[List[list1[[s]],list2[[t]]],
     {s,1,llist1},{t,1,llist2}],1];

wedgelist=Table[List[zz],{zz,1,Length[newlist]}];

For[cc=1, cc<=Length[newlist], cc++,
   For[dd=2, dd<=Length[newlist[[cc,1]]], dd++,
      For[ee=2, ee<=Length[newlist[[cc,2]]], ee++,
            position=(newlist[[cc,1,dd]]-1)*llist2  +
                                        newlist[[cc,2,ee]];

            AppendTo[wedgelist[[cc]],position];
         ]
      ]
  ];

wedgelist
)

(*******************     P(4,1,2)^K(2,3)     **********************)
wedge[powergraph[4,1,2],completeBipartite[2,3],6,2]
Clear[f,ff,g,BBcoord,WWcoord];
OpenWrite["Tvoya Mat:Potanka:P(4,1,2)^K(2,3)",
            FormatType->OutputForm];
WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)","P(4,1,2)^K(2,3)\n"];
Write["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)",Length[wedgelist]];
For[f=1,  f<=Length[wedgelist],f++,
        WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)",-f];
        For[g=2,  g<=Length[wedgelist[[f]]],  g++,
           WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)","  ",
```

```
                          wedgelist[[f,g]]
              ];
           If[f==Length[wedgelist]  &&  g==Length[wedgelist[[f]]],
              WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)","  ",0]];
          ];
        Write["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)"];

  ];

Write["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)",
           "&coordinates  of  vertices:"];

BBcoord=Floor[N[350/Length[newlistB]]];
WWcoord=Floor[N[350/Length[newlistW]]];

For[ff=1,  ff<=Length[wedgelist],ff++,
        WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)",-ff];

        If[ff<=  Length[newlistB],
           WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)","  ",430],
           WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)","  ",130]
         ];

        If[ff<=  Length[newlistB],
           WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)","  ",
                             ff*BBcoord],
           WriteString["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)","  ",
                             (ff-Length[newlistB])*WWcoord]
         ];

        Write["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)"]

         ];

Close["Tvoya  Mat:Potanka:P(4,1,2)^K(2,3)"]
```

Appendix C

Wedge Product Graphs

Graph	# of top vertices	Degree of top vertices	# of bot vertices	Degree of bot. vertices	Total # of vertices	Total # edges	Order of the group	Distance
P(4,1,2)^K(2,3)	12	6	12	6	24	72	$2^{13}3^{15}$	≤3
P(5,1,2)^K(1,2)	10	4	10	4	20	40	3840	≤3
P(5,1,3)^K(1,2)	10	6	10	6	20	60	3840	≤3
P(6,1,2)^K(2,5)	30	10	30	10	60	300	$2^{37}3^{8}5^{7}$	≤3
P(6,1,3)^K(3,10)	60	30	60	30	120	1800	$2^{72}3^{46}5^{13}7^{6}$	≤3
P(6,1,4)^K(2,5)	30	20	30	20	60	600		≤3
P(6,2,3)^K(3,4)	60	12	60	12	120	720		≤5
P(7,1,2)^K(1,3)	21	6	21	6	42	126	$2^{11}3^{9}5^{1}7^{1}$	≤3
P(7,1,3)^K(1,5)	35	15	35	15	70	525	$2^{25}3^{9}5^{8}7^{1}$	≤3
P(7,1,4)^K(1,5)	35	20	35	20	70	700		≤3
P(7,1,5)^K(1,3)	21	15	21	15	42	315	$2^{11}3^{9}5^{1}7^{1}$	≤3
P(7,2,3)^K(3,5)	105	15	105	15	210	1575		≤5
P(7,2,4)^K(3,5)	105	30	105	30	210	3050		≤3
P(8,1,2)^K(2,7)	56	14	56	14	112	784		≤3
P(8,1,3)^K(1,7)	56	21	56	21	112	1176		≤3
P(8,1,4)^K(4,35)								≤3
P(8,1,5)^K(1,7)	56	35	56	35	112	1960		≤3
P(8,1,6)^K(2,7)	56	42	56	42	112	2352		≤3
P(8,2,3)^K(1,2)	56	6	56	6	112	336	$2^{35}3^{2}5^{1}7^{1}$	≤5
P(8,2,4)^K(2,5)	140	30	140	30	280	4200		≤3
P(8,2,5)^K(1,2)	56	20	56	20	112	1120		≤3
P(8,3,4)^K(4,5)	280	20	280	20	560	5600		≤7

Appendix D

Comparison of P(N,1,b) to Distance3(P(N,1,N-b))

Graph	# of top vertices	Degree of top vertices	# of bot vertices	Degree of bot vertices	Total # of vertices	Total # of edges	Order of the group
P(7,1,5)	21	5	7	15	28	105	5040
Distance3(P(7,1,2))	21	5	7	15	28	105	5040
P(7,1,4)	35	4	7	20	42	140	5040
Distance3(P(7,1,3))	35	4	7	20	42	140	5040
P(7,1,3)	35	3	7	15	42	105	5040
Distance3(P(7,1,4))	35	3	7	15	42	105	5040
P(7,1,2)	21	2	7	6	28	42	5040
Distance3(P(7,1,5))	21	2	7	6	28	42	5040
P(8,1,5)	56	5	8	35	64	280	40320
Distance3(P(8,1,3)	56	5	8	35	64	280	40320
P(8,1,6)	28	6	8	21	36	168	40320
Distance3(P(8,1,2))	28	6	8	21	36	168	40320
P(7,1,2)^K(1,3)	21	6	21	6	42	126	$2^{11}3^{9}5^{1}7^{1}$
Distance3(P(7,1,5)^K(1,3)	21	6	21	6	42	126	$2^{11}3^{9}5^{1}7^{1}$
P(5,1,2)^K(1,2)	10	4	10	4	20	40	3840
Distance3(P(5,1,3)^K(1,2)	10	4	10	4	20	40	3840

Appendix E

Distance-3 of Wedge Product Graphs

Graph	# of top vertices	Degree of top vertices	# of bot vertices	Degree of bot. vertices	Total # of vertices	Total # edges	Order of the group	Distance
Distance3(P(4,1,2)^K(2,3))	12	6	12	6	24	24	$2^{13}3^5$	≤3
Distance3(P(5,1,2)^K(1,2)	10	6	10	6	20	60	3840	≤3
Distance3(P(5,1,3)^K(1,2)	10	4	10	4	20	40	3840	≤3
Distance3(P(6,1,2)^K(2,5)	30	20	30	20	60	600		≤3
Distance3(P(6,1,3)^K(3,10)	60	30	60	30	120	1800		≤3
Distance3(P(6,1,4)^K(2,5)	30	10	30	10	60	300	$2^{37}3^{8}5^{7}$	≤3
Distance3(P(6,2,3)^K(3,4)	60	36	60	36	120	2160		≤5
Distance3(P(7,1,2)^K(1,3)	21	15	21	15	42	315	$2^{11}3^{9}5^{1}7^{1}$	≤3
Distance3(P(7,1,3)^K(1,5)	35	20	35	20	70	700		≤3
Distance3(P(7,1,4)^K(1,5)	35		35		70			
Distance3(P(7,1,5)^K(1,3)	21	6	21	6	42	126	$2^{11}3^{9}5^{1}7^{1}$	≤3
Distance3(P(7,2,3)^K(3,5)	105		105		210			
Distance3(P(7,2,4)^K(3,5)	105		105		210			
Distance3(P(8,1,2)^K(2,7)	56	42	56	42	112	2352		≤3
Distance3(P(8,1,3)^K(1,7)	56	35	56	35	112	1960		≤3
Distance3(P(8,1,4)^K(4,35)								
Distance3(P(8,1,5)^K(1,7)	56	21	56	21	112	1176		≤3
Distance3(P(8,1,6)^K(2,7)	56	14	56	14	112	784		≤3
Distance3(P(8,2,3)^K(1,2)	56	30	56	30	112	1680		≤5
Distance3(P(8,2,4)^K(2,5)	140		140		280			
Distance3(P(8,2,5)^K(1,2)	56	36	56	36	112	2016		≤3
Distance3(P(8,3,4)^K(4,5)	280		280		560			