LEAH WRENN BERMAN
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SPIDERGRAPHS

0. ABSTRACT

This paper will discuss a class of graphs called spidergraphs. etc. etc. etc.

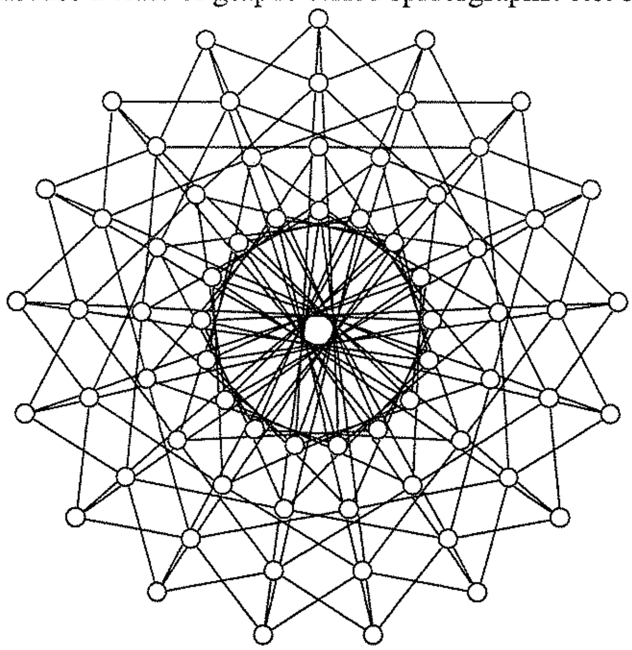

FIGURE 1. An example of a spidergraph: S(4, 17; 1,2,4,8)

1. INTRODUCTION: WHAT IS A SPIDERGRAPH?

Like all graphs, spidergraphs are formed by connecting a number of vertices by a number of edges in a certain pattern. Spidergraphs have additional structure caused by the organization of the vertices into rings and slots. Vertices in one ring are connected to vertices in the next ring in, but vertices in the same ring are not connected to each other (see figure 1). More generally, we say that a spidergraph is formed of k rings with N vertices per ring, where the N vertices are thought of as being located in certain slots. An individual vertex is labeled (i, j), where i is the ring in which it is located and j is the slot it occupies in that ring. The connections between the rings are governed by the **connection sequence** $a_0, a_1, \ldots a_{k-1}$, where vertex (i, j) is connected to vertices $(i+1 \pmod k, j+a_i \pmod N)$ and $(i+1 \pmod k, j-a_i \pmod N)$. From now on, these connections will be abbreviated $(i+1, j+a_i)$ and $(i+1, j-a_i)$; the appropriate moduli will be assumed. Spidergraphs are denoted $S(k,N;\ a_0,\ a_1,\ \ldots\ a_{k-1})$.

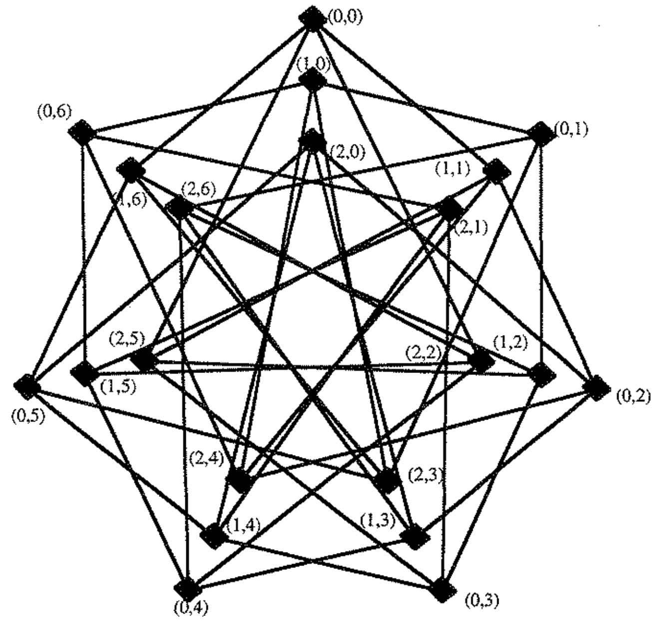

FIGURE 2: S(3,7; 1,2,4)

Consider the spidergraphs S(3,7; 1,2,4) and S(3,7; 1,3,9). Just from their names, it seems reasonable that they are 2 different spidergraphs. After all, although they have the same number of vertices and the same division into rings, they are connected by different connection sequences. Clearly, $s_1 = (1,2,4)$ and $s_2 = (1,3,9)$ are different sequences. However, since ring i is always connected to ring $i+1 \pmod{k}$, the connection sequence only affects the *slot* of the vertex. Thus, the connection sequence may be reduced mod N. After such reduction, $s_1 = (1,2,4)$ and $s_2 = (1,3,2)$. Since a_i and $-a_i$ form the same two connections as $-a_i$ and $-(-a_i)$, if $a_i > \left\lfloor \frac{N}{2} \right\rfloor$, we may write a_i as $(a_i - N)$, which is equivalent to $-(a_i - N)$ in the connection sequence. Thus, $s_1 = (1,2,3)$. It is now clear that s_1 and s_2 are much more similar that was originally thought. All that remains to show that S(3,7; 1,2,4) and S(3,7; 1,3,9) are, in fact, isomorphic is to remark that the order of the sequence—that is, the order in which the rings are placed when constructing the graph—is irrelevant. The canonical order of a connection sequence is in ascending numerical order, but from time to time, to emphasize different aspects of spidergraphs, sequences may be ordered differently.

In addition to reducing a connection sequence mod N and rewriting elements of the sequence so that they are between 1 and $\left\lfloor \frac{N}{2} \right\rfloor$, there are a few other legitimate manipulations for elements of the connection sequence. If the entire sequence is multiplied by some integer t, where $\gcd(t, N) = 1$, then the sequence remains the same, since multiplication by an integer which is relatively prime is an automorphism.

Within a spidergraph, we already are comfortable with the idea of rings and slots. There is one more piece of terminology which is useful when discussing spidergraphs. A **webbing** is all edges connecting one ring with another.

An important class of spidergraphs are known as **power spidergraphs.** Power spidergraphs have connection sequences of the form $(a^0, a^1, a^2, \ldots, a^{k-1})$, where a is an integer and $N \mid a^k \pm 1$; that is, $a^k \equiv \mp 1 (\bmod N)$.

2. Symmetries and Transitivity

2.1 A few definitions

Definition: A *symmetry (automorphism)* of a graph S is a permutation σ of the vertices of S so that for every edge $\{x,y\}$, $\sigma(\{x,y\})$ is also an edge of the graph. That is, σ preserves adjacencies.

It is well known that the set of all symmetries of a graph forms a group under composition. We will refer to this group as $G = G(S) = \mathrm{Aut}(S)$.

Another important concept is that of transitivity.

Definition: A graph S is *vertex-transitive* if G acts transitively on the vertices. That is, there exists some permutations in G which send any vertex to any other vertex in the graph. Likewise, S is edge transitive if G acts transitively on the edges.

Edges can be thought of as consisting of two darts, one for each possible orientation which may be induced upon that edge (see figure 3).

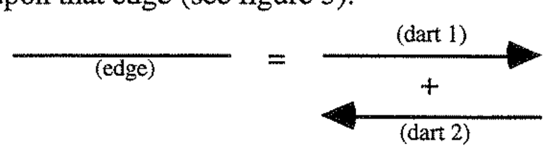

FIGURE 3: DARTS

With that in mind, *S* is *dart-transitive* if G acts transitively on the darts.

Definition: A graph *S* is *semi-transitive* if there exists a subgroup H of G which acts transitively on the vertices and edges, but not on the darts. *S* is *strictly semi-transitive* if H=G.

2.2 IMPORTANT SYMMETRIES

Every spidergraph, whether or not it is a power spidergraph, has two important symmetries: a rotation symmetry and a reflection symmetry. The first, the rotation, is, mathematically:

$$\rho: V(S) \to V(S) \text{ via } \rho(i,j) = (i, j+1).$$

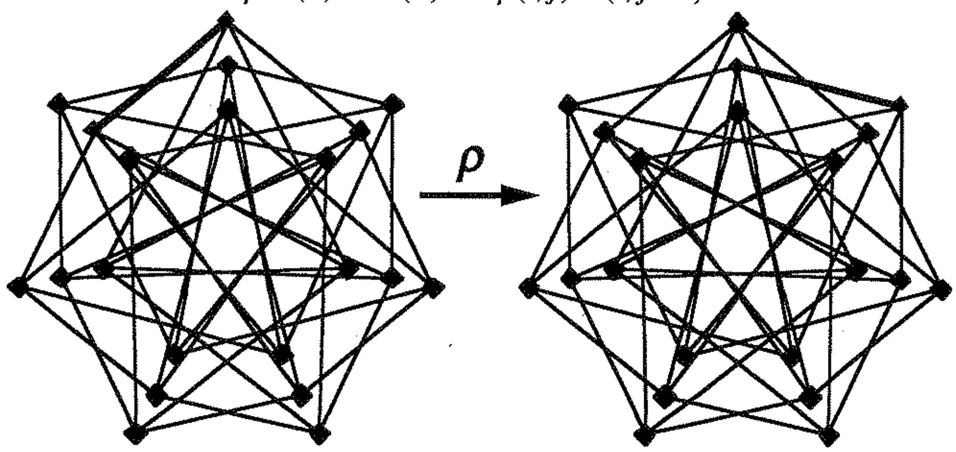

FIGURE 4: APPLICATION OF ρ TO S(3,7; 1,2,4)

The second symmetry of every spidergraph is:

$$\chi: V(S) \to V(S) \text{ via } \chi(i,j) = (i,-j).$$

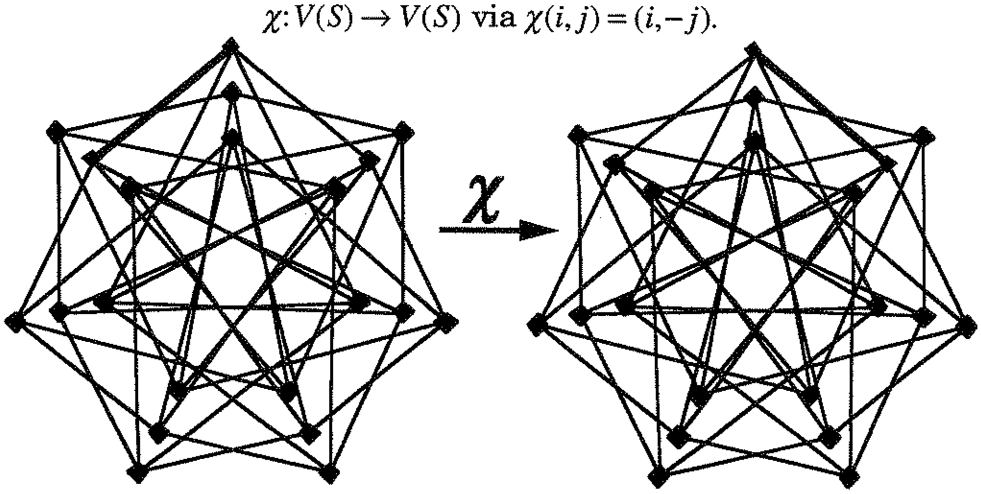

FIGURE 5: APPLICATION OF χ TO S(3,7; 1,2,4)

By inspection, it is clear that these two must occur in every spidergraph (see Figures 4 and 5). It is also very easy to prove algebraically that these two permutations are symmetries. It turns out that the group generated by ρ and χ is the dihedral group D_N, where N is the number of vertices per ring. Thus, the order of G for any spidergraph is at least $2N$. Intuitively, this makes sense, since you can rotate the graph around in one direction N times before you get back to where you started (corresponding to application of ρ N times) and there are N axes of reflection (each corresponding to application of $\rho^q\chi$ for some integer q). It is easy to see that ρ and χ are enough to send any vertex to any other vertex in a ring and any edge to any other edge in a webbing.

For power spidergraphs, there is one other important symmetry:

$$\sigma: V(S) \to V(S) \text{ via } \sigma(i,j) = (i+1, ja).$$

Note that this assumes a connection sequence of $(a^0, a^1, a^2, \dots, a^{k-1})$. Also, unlike ρ and χ, σ sends vertices in one ring to vertices in a different ring. The order of the group generated by σ is clearly k, since σ can act on k different types of elements.

Lemma: σ is a symmetry of the power spidergraph $S = S(k,N; a^0, a^1, a^2, \dots, a^{k-1})$.

Proof: Choose an edge in S, $\{(i,j), (i+1, j+a^i)\}$. By applying σ, we see that $\sigma(\{(i,j),(i+1,j+a^i)\}) = \{(i+1,ja),(i+2,a(j+a^i))\}$. But since S is a power spidergraph, $(i+1,ja)$ is connected to $(i+2, ja+a^{i+1})$. However, $(i+2,a(j+a^i)) = (i+2, ja+a^{i+1})$, so $\sigma(\{(i,j),(i+1,j+a^i)\}) = \{(i+1,ja),(i+2,a(j+a^i))\}$ is an edge in S, making σ a symmetry.

Since σ moves a vertex from one ring to another, and ρ and χ send that vertex to any other vertex in its ring, σ, ρ and χ, in various combinations, are sufficient to ensure vertex-transitivity. That is, the group $<\sigma, \rho, \chi>$ acts transitively on the vertices of S. Likewise, it is easy to see that the group $<\sigma, \rho, \chi>$ acts transitively on the edges of S.

No combination of ρ or χ will send a dart of one orientation to a dart of the opposite orientation. This can be seen by inducing an orientation of one edge and then applying ρ and χ—both symmetries preserve orientation. Likewise, σ preserves orientation: although applying σ to an oriented edge (e.g. out to in) in the webbing from ring k-2 to ring k-1 will yield an edge which appears to have an opposite orientation (in to out, say), except in unusual circumstances (if the connection sequence is all the same number), that in-to-out oriented edge will not be mappable, using σ, ρ and χ, to the original edge. Thus, unless the connection sequence is equivalent to (1,1, ... , 1), the group $<\sigma, \rho, \chi>$ does not act transitively on the darts of S.

Thus, for any power spidergraph $S = S(k,N; a^0, a^1, a^2, \ldots, a^{k-1})$ with a non-trivial connection sequence (i.e. $a \neq 1$), the group $<\sigma, \rho, \chi>$ acts transitively on the edges and vertices, but not on the darts, of S. Moreover, $<\sigma, \rho, \chi>$ is a subgroup of G = Aut(S); its order is $2kN$. Thus, every (non-trivial) power spidergraph is semi-transitive.

The obvious next question to ask is what makes a power spidergraph strictly semi-transitive. Unfortunately, this is still an open question. Consider the following table (Table 1):

Spidergraph *S*	**| Aut(*S*) |**
power spidergraphs	
S(3,7; 1,2,4)	$((2\cdot7)\cdot3)\cdot2$
S(3,9; 1,2,4)	$(2\cdot9)\cdot3$
S(3,14; 1,3,9)	$((2\cdot14)\cdot3)\cdot2^4$
S(3,26; 1,3,9)	$(2\cdot26)\cdot3$
S(4,8; 1,1,1,1)	$((2\cdot8)\cdot4)\cdot2^{19}$
S(4,41; 1,3,9,27)	$(2\cdot41)\cdot4$
S(5,33; 1,2,4,8,16)	$(2\cdot33)\cdot5$
non-power spidergraphs	
S(3,11; 1,2,4)	$2\cdot11$
S(3,12; 1,3,9)	$(2\cdot12)\cdot2^8$
S(3,14; 1,3,2)	$(2\cdot14)\cdot2^3\cdot3^2\cdot7$
S(4,41; 1,4,16,64)	$2\cdot41$

TABLE 1: SPIDERGRAPHS AND ORDERS OF THEIR AUTOMORPHISM GROUPS

The figures for the orders of the various automorphism groups were generated by a computer program.[1] Some of the power spidergraphs, such as S(3,9; 1,2,4), are strictly semi-transitive: the order of $<\sigma, \rho, \chi> = 2\cdot3\cdot9$, and $|$ Aut(S(3,9; 1,2,4)) $| = 2\cdot3\cdot9$ also. Since $<\sigma, \rho, \chi> \leq$ Aut(S(3,9; 1,2,4)), the two groups are equal. On the other hand, S(3,7; 1,2,4) is *not* strictly semi-transitive, and it is not clear why.

Other spidergraphs in this table have strange automorphism groups, also. For instance, consider the group for S(4,8; 1,1,1,1). Admittedly, this graph is not strictly semi-transitive, but it's group is unexpectedly enormous: $|$ Aut(S(4,8; 1,1,1,1)) $| = 2^{25}$. That's 33554432 different automorphisms; we know of 32 of those. A slightly smaller, but equally strange example is S(3,14; 1,3,2), with an automorphism group of order $2^5\cdot3^2\cdot7^2$. That's a lot of automorphisms for a graph which is not even a power graph!

To gain insight into why these graphs have automorphism groups which are so large, consider the graph of S(3,14; 1,3,2) (see Figure 6):

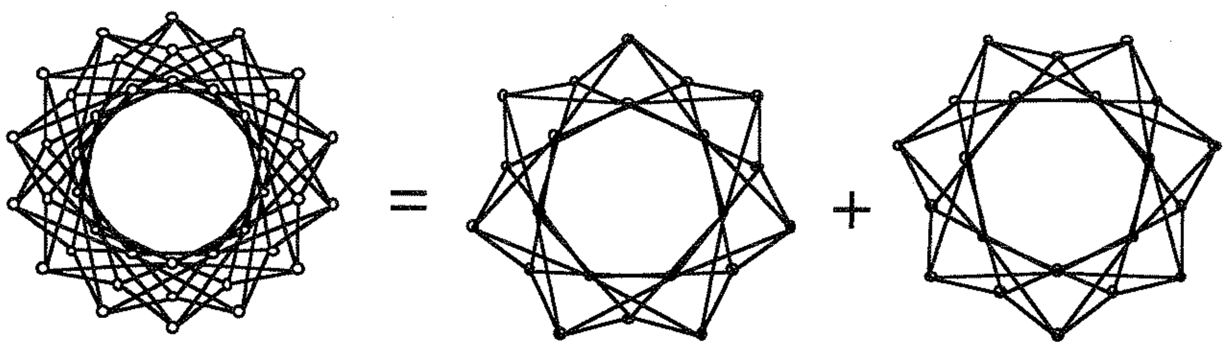

FIGURE 6: S(3,14; 1,3,2) AND ITS 2 COMPONENTS

Since S(3,14; 1,3,2) is a disconnected graph, we expect that it would have a quite large automorphism group. After all, each of the 2 components has at least the dihedral symmetries, plus an extra symmetry sending one component to the other (since one component is isomorphic to the other). If a component itself has a fair number of symmetries (as in the case of S(4,8; 1,1,1,1)), the number of symmetries of the entire (disconnected) spidergraph quickly becomes very large.

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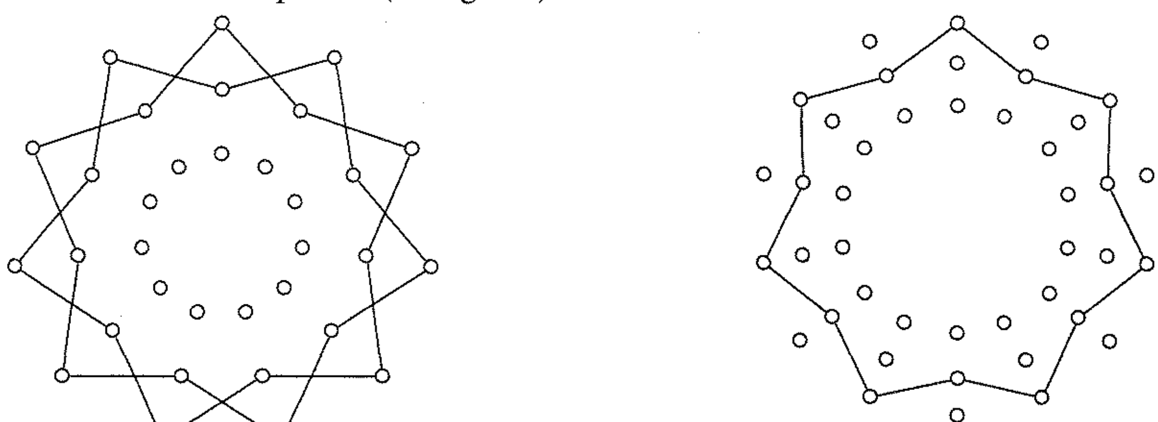
[1] *Groups & Graphs 2. 3,* written by William Kocay, University of Manitoba.

The question, of course, is to find out when spidergraphs are disconnected; since it is the connection sequence which controls the placement of edges, that is equivalent to determining what characteristics the connection sequence must have for the spidergraph to be disconnected.

Before we decide what makes a spidergraph be disconnected, it is helpful to consider the webbings of the spidergraph. A webbing may be connected; that is, it is possible to travel from a vertex back to that same vertex, hitting all vertices in both rings of the webbing and all edges of the webbing. On the other hand, a webbing may be disconnected into some number of components (see figure 7).



S(3,11; 1,—,—)

S(3,14; 1,—,—)
(one component)

FIGURE 7: THE TWO TYPES OF WEBBINGS

Insert lots and lots of stuff on connectivity

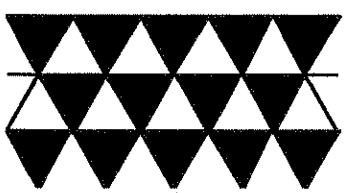
4. TORI, MAPS, MEDIAL GRAPHS AND SPIDERGRAPHS

Although determining whether a spidergraph is connected or disconnected is an important thing to know about a spidergraph, and while knowing that a spidergraph is disconnected does give important insight into the order of that graph's automorphism group, we are still interested in looking at the symmetries of a *connected* spidergraph. To investigate these further, we looked at the medial graphs of certain maps on the torus.

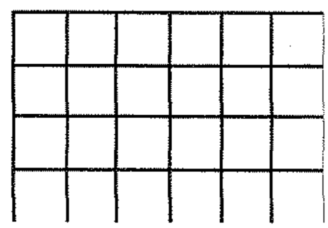
4.1 MAPS ON THE TORUS

Definition: A map *M* is an embedding of a graph in a surface so as to divide it into simply connected regions.

Several years ago, H.S.M. Coxeter determined three important classes of maps on the torus: $\{3,6\}_{b,c}$, its dual, $\{6,3\}_{b,c}$, and $\{4,4\}_{b,c}$. All three of these maps are formed from the tessellation of the Euclidean plane with regular polygons. The $\{3,6\}_{b,c}$ is formed by tessellating equilateral triangles (that is, regular 3-gons) so that they meet 6 at a vertex; likewise, $\{6,3\}_{b,c}$ uses tessellated hexagons meeting three at a vertex and $\{4,4\}_{b,c}$ uses tessellated squares (figure 8). Since $\{6,3\}_{b,c}$ is the dual of $\{3,6\}_{b,c}$, I will concentrate on $\{3,6\}_{b,c}$ without worrying that there is different information to be gained from considering $\{6,3\}_{b,c}$.


{3,6}


{6,3}


{4,4}

Thus, the {x,y} has been explained for these maps on the torus. However, merely tessellating the Euclidean plane is not sufficient to produce a map on a torus. The map on the torus occurs by defining a parallelogram, whose dimensions are determined by the values of b and c, on the tessellated plane and identifying certain edges and vertices: the top edge with the bottom edge, the left edge with the right edge, and all four vertices on the corners of the parallelogram with each other (figure 9). Such identification creates a finite map on the torus with certain characteristics (table 2):

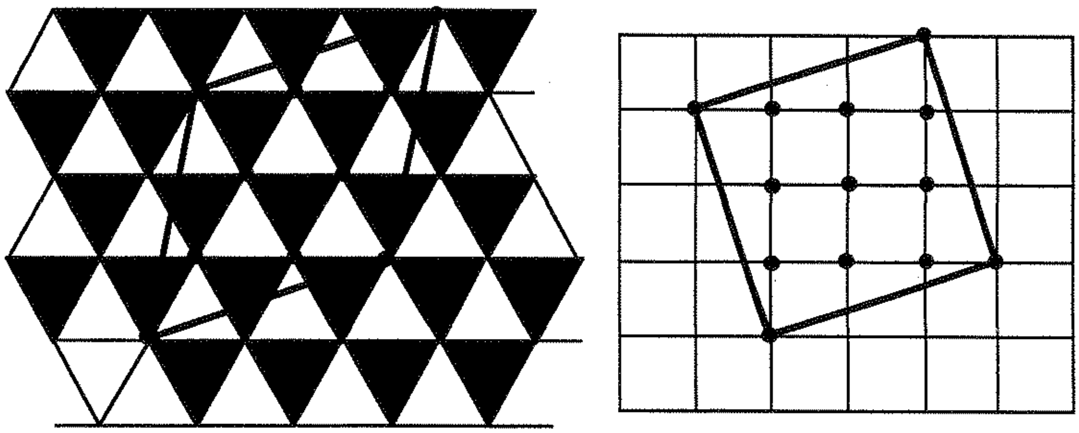


FIGURE 9: $\{3,6\}_{2,1}$ AND $\{4,4\}_{3,1}$

Map	# of vertices = D	# of edges
$\{3,6\}_{b,c}$	$b^2 + bc + c^2$	$3D$
$\{4,4\}_{b,c}$	$b^2 + c^2$	$2D$

TABLE 2: CHARACTERISTICS OF $\{3,6\}_{b,c}$ AND $\{4,4\}_{b,c}$

The next map on the torus which is important to our exploration of spidergraphs is called the **medial graph**. It is formed from a map Q in the following way: a vertex in the medial graph M(Q) is formed at the midpoint of every edge in Q, and two vertices in M(Q) are connected by an edge if and only if the edges from which they were formed in Q share a face (see Figure 10). Since every edge of Q contributes one vertex to M(Q), it is clear that there are three times as many vertices in M(Q) as in Q.

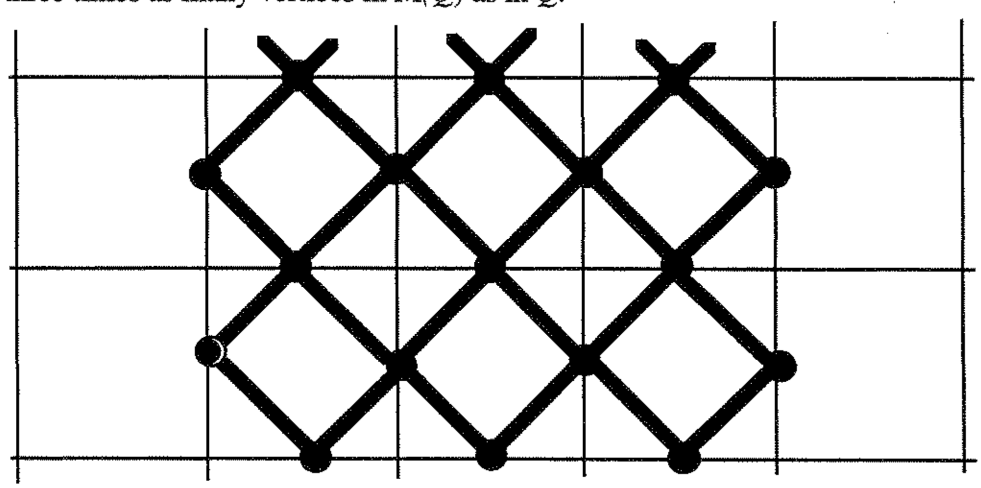


FIGURE 10: PART OF A MEDIAL GRAPH ON $\{4,4\}_{b,c}$

Why are these medial graphs important? It turns out that certain of these graphs may be labeled quite easily as spidergraphs. A nice small example is $\{3,6\}_{2,1,}$, whose medial graph is S(3,7; 1,2,4) (figure 11).

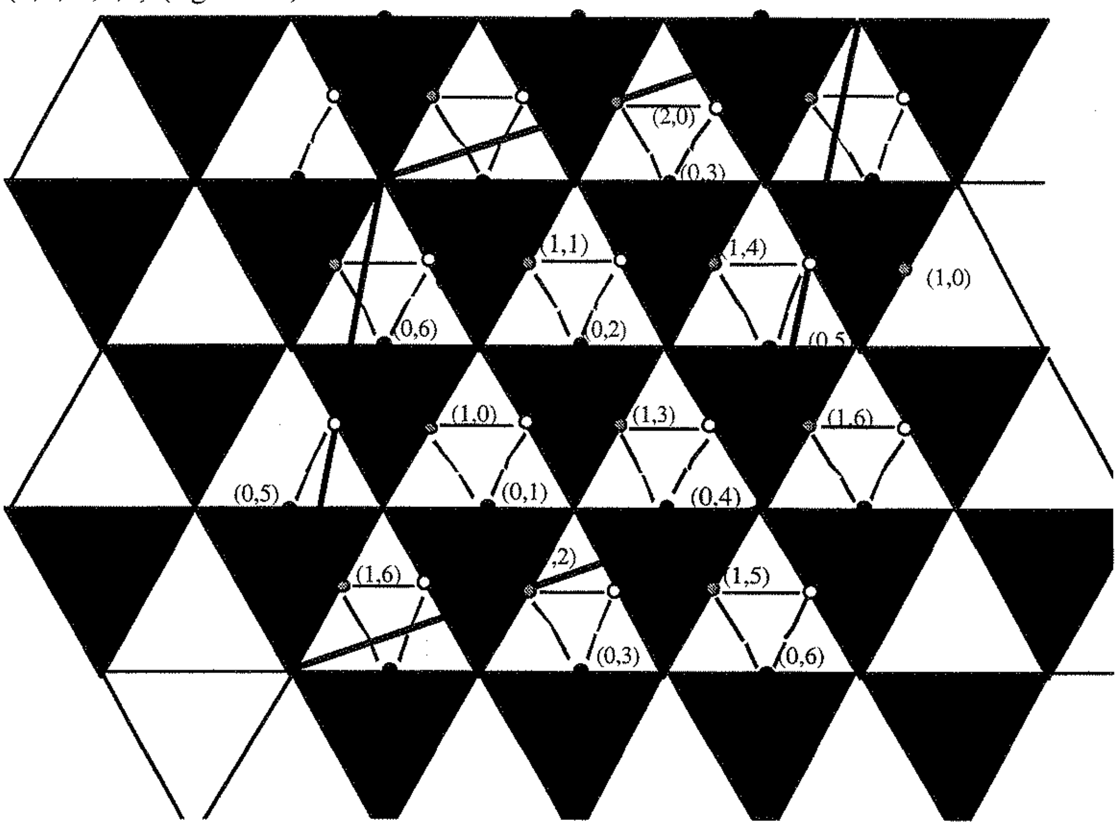


FIGURE 11: M($\{3,6\}_{2,1}$ LABELED AS S(3,7; 1,2,4)

In fact, there is an entire family of medial graphs on both $\{3,6\}_{b,c}$ and $\{4,4\}_{b,c}$ which may be labeled as spidergraphs.

SPIDERGRAPHS ON $\{3,6\}_{b,c}$:

important big hard-to-draw diagram

The above diagram assumes the following hypotheses; some of them are apparent form the partial labeling given:

1) As above, $D = b^2 + bc + c^2$ = the number of vertices per color. There are $3D$ vertices in the medial graph, divided into three parts, corresponding to three rings, equally.

2) The medial graph may be shifted or rotated; a horizontal shift of one unit shall be called a distance of x; thus, the vertex (0,0) will be shifted to the vertex (0, x).

3) A diagonal shift of one unit corresponds to a distance of 2; that is, (0,0) shifts to (0,2).

4) gcd(b,c) = 1; this ensures that the number of vertices per color (D) is odd.

5) $a_0 = 1$.

From the diagram, it is possible to determine what the connection sequence is: a point in ring 1 lies between 2 points in ring 0; if we are to label this medial graph as a spidergraph, that point must lie 1 ring in and "half" the distance between. Of course, the distance between the 2 points in ring 0 may not be even; however, since D is odd, by one of the assumptions, we know that $2^{-1} \bmod D$ exists. Thus, the connection jump is the difference in the j-values, multiplied by 2^{-1}.

By hypothesis, $a_0 = 1$.

Using the ——— and ——— colored vertices, $a_1 = 2^{-1}((1+x)-1) = 2^{-1}x$. Likewise, $a_2 = 2^{-1}(x - 2) = 2^{-1}x - 1$.

Again looking at the diagram, it is clear that $x(b+c) \equiv 2c \bmod D$. Assume $x = s(b+c)$ for some integer s. Then

$$s(b+c)^2 =$$
$$s(b^2+2bc+c^2) =$$
$$sb^2+sbc+sc^2+sbc =$$
$$s(b^2+bc+c^2)+sbc \equiv 2c \bmod D.$$

But $D = b^2 + bc + c^2$, so $s(b^2+bc+c^2) \equiv 0 \bmod D$. Then $sbc \equiv 2c \bmod D$. Since gcd $(b,c) = 1$, it is very easy to show that gcd(b,D) = 1 and gcd(c,D) = 1. Thus $sb \equiv 2 \bmod D$ and $s \equiv 2b^{-1} \bmod D$.

Thus, $x = 2b^{-1}(b+c)$. Using this value in the connection sequence, $a_0 = 1$, $a_1 = 1 + b^{-1}c$, and $a_2 = b^{-1}c$.

Thus, the spidergraph formed by this method of labeling the medial graph is S(3,D; $1, 1+b^{-1}c, b^{-1}c$).

Moreover, consider $(a_1)^2 = (1+b^{-1}c)^2$.

$(1+b^{-1}c)^2 \bmod D =$

$(b^{-1})^2 c^2 + b^{-1}c + 1 + b^{-1}c \bmod D =$

$(b^{-1})^2 c^2 + b^{-1}c(bb^{-1}) + 1(b^2(b^{-1})^2) + b^{-1}c \bmod D =$

$(b^{-1})^2(c^2 + bc + b^2) + b^{-1}c \bmod D =$

$b^{-1}c \bmod D = a_2$.

Also, $(a_1)^3 = (1+b^{-1}c)(b^{-1}c)$. Then

$(1+b^{-1}c)(b^{-1}c) \bmod D =$

$(b^{-1})^2 c^2 + b^{-1}c \bmod D$. But $D = b^2 + bc + c^2$, so $c^2 = -b^2 - bc$.

Then $(b^{-1})^2 c^2 + b^{-1}c = (b^{-1})^2(-b^2 - bc) + b^{-1}c =$

$-1 - b^{-1}c + b^{-1}c =$

$-1 \bmod D$.

That is, we have a spidergraph whose connection sequence is of the form (a^0, a, a^2) and $a^3 = -1$: this is the precise form of a power spidergraph!

insert similar section on {4,4}'s

insert stuff on further research opportunities

insert conclusion

APPENDIX 1

MATHEMATICA CODE TO PRODUCE SPIDERGRAPHS USING *GROUPS & GRAPHS*

```
Spidergraph[title_, n_, sequ_] :=

    (
    (* create connections *)
     k:=Length[sequ];
     graphlist = {"&Graph",title, k n};
     j=0;
     While[j<(k n),
       ring=Ceiling[(j+1)/n];

       value1=Mod[Mod[(j+sequ[[ring]]), n] + 1 +ring*n, k n];
       value2=Mod[Mod[(j-sequ[[ring]]), n] + 1 +ring*n, k n];

       AppendTo[graphlist, {-(j+1),
               If[value1==0, k n, value1],
               If[value2==0, k n, value2]}];
       j++;];

      AppendTo[graphlist, 0];
      AppendTo[graphlist, "&PtNames"];
      AppendTo[graphlist, "&Coordinates of vertices:"];

      (* create coordinates *)

      j=1;
      ringdist=198;
      ctrx=300;
      ctry=200;

      While[j<=(k n),
                     q=0;
                     theta=Pi/2;
                     While[q<n,

                            xdist=ctrx + ringdist (Cos[theta]);
                            ydist=ctry - ringdist (Sin[theta]);

                            AppendTo[graphlist, {-(j), Floor[N[xdist]],
                            Floor[N[ydist]]}];

                            q++;
                            j++;
                            theta=theta+(2 Pi/n);];

                     ringdist=ringdist-(180/k);];


     TableForm[graphlist])
```

```
components[title_, n_, sequ_] :=

    (
    k:=Length[sequ];

    s=1;
    sumseq=0;
    While[s<=k,
      sumseq=sumseq+sequ[[s]];
      s++;];

    (* create connections *)

     graphlist = {"&Graph",title, k n};
     j=0;
     While[j<(k n),
       ring=Ceiling[(j+1)/n];

       value1=Mod[Mod[(j+sequ[[ring]]), n] + 1 +ring*n, k n];
       value2=Mod[Mod[(j-sequ[[ring]]), n] + 1 +ring*n, k n];

       AppendTo[graphlist, {-(j+1),
               If[value1==0, k n, value1],
               If[value2==0, k n, value2]}];
       j++;];

      AppendTo[graphlist, 0];

      (* create coordinates *)
      coordlist={"&PtNames", "&Coordinates of vertices:"};
      j=1;
      ringdist=198;
      ctrx=300;
      ctry=200;

      While[j<=(k n),
                     q=0;
                     theta=Pi/2;
                     While[q<n,

                            xdist=ctrx + ringdist (Cos[theta]);
                            ydist=ctry - ringdist (Sin[theta]);

                            AppendTo[coordlist, {-(j), Floor[N[xdist]],
                            Floor[N[ydist]]}];

                            q++;
                            j++;
                            theta=theta+(2 Pi/n);];

                     ringdist=ringdist-(180/k);];

  (* is it connected? *)
```

```
If[ OddQ[n] || OddQ[sumseq],

  graphlist = Join[graphlist, coordlist];
  Print["This is a connected graph.\n"];
  TableForm[graphlist],


        (* component connections(in the if statement) *)
        componentlist={"&Graph",title, k n};

        j=1;

        While[j<= k n,


            While[j<=n,

                AppendTo[componentlist, graphlist[[j+3]] ];
                j=j+2 sequ[[1]];

                ];


            If[MemberQ[ Flatten[Drop[componentlist,3]], j ],
                AppendTo[componentlist, graphlist[[j+3]] ];
                j++;,
                j++;];
            ];

        AppendTo[componentlist, 0];

        componentlist = Join[componentlist, coordlist];

  TableForm[componentlist]

]

  )
```