Time-Delayed Iterated Maps

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1. Introduction

A mathematical model has been used to study two interacting insect populations, using the family of iterated maps given by: $x \mapsto ax^3 + (1-a)x$, or $x_{n+1} = ax_n^3 + (1-a)x_n$, which exists for $0 < a < 4$.[1] Our intention is to analyze this equation, given a time delay, so that now our map is $(x,y) \mapsto (y, y(ax^2 + (1-a)))$, or, $\begin{cases} x_{n+1} = y_n \\ y_{n+1} = y_n(ax_n^2 + (1-a)) \end{cases}$ for $0 < a < 4$.[2] To simplify this system to a single equation, we can also consider it as: $x_{n+1} = x_n(ax_{n-1}^2 + (1-a))$. To discuss this family of maps, we must first begin with a review of dynamical systems, specifically, for iterated maps.

An iterated map takes the general form $X_{n+1} = f(X_n)$. Unlike a differential equation, an iterated map does not map points continuously along a line. Rather, it jumps discretely from point to point. In our diagrams, the lines we see are showing overall general direction of the iterations, not a continuous set of points. When we see limit cycles, they are not continuous either. In reality, they are many, closely spaced, discrete points.

Note: for future reference, the notation, $x \mapsto ax$ is used to say $x_{n+1} = ax_n$.
Likewise, in two dimensions, $(x,y) \mapsto (ay, bx + y)$ would represent:

$$x_{n+1} = ay_n$$
$$y_{n+1} = bx_n + y_n$$

Fixed Points: After several iterations, a solution may eventually settle down to one value, which is the result of each subsequent iteration. In other words, $X_{n+1} = X_n$, or, $f(X_n) = X_n$. The value that satisfies this equation is called a **fixed point** of the equation. As an example, we consider the Logistic Map: $x_{n+1} = \mu x_n(1 - x_n)$, a famous family of maps that vary with the parameter, μ . Setting $X_{n+1} = X_n$ and solving, we find that for the logistic map, the fixed points occur at $x = 0$ and $x = \frac{\mu - 1}{\mu}$.

2-Cycles: After several iterations, instead of settling down to one value, the solution might jump back and forth between two values from iteration to iteration. In other words, $X_{n+2} = X_n$, or, $f(f(X_n)) = X_n$. These two points are called a **2-cycle** of the equation.

For the logistic map, 2-cycles occur at $x = \frac{\mu + 1 \pm \sqrt{(1+\mu)(\mu-3)}}{2\mu}$ for $\mu > 3$.

n-Cycles: In the same way that 2-cycles occur, we can often find other numbers of n-cycles. 3-cycles, for example, follow the same pattern of behavior as 2-cycles, except with three points instead. For example, in the simple 2-dimensional system:

$$\begin{cases} X_{n+1} = Y_n \\ Y_{n+1} = -X_n - Y_n \end{cases}$$

the points $(1,0) \mapsto (0,-1) \mapsto (-1,1) \mapsto (1,0)$ form a three cycle of the map.

Stabilities of fixed points and 2-cycles: While the existence of fixed points and n-cycles is important, more significant are the stabilities of these values. By stability, we mean that if we begin with an initial point that is close to a stable fixed point, we will be drawn in toward it, asymptotically. If we begin close to an unstable fixed point, however, we will be thrown away from it. The only way that we can stay at an unstable fixed point is if we are already on it exactly.

Theorem: [3] Let x_0 be a fixed point of f. Then x_0 is stable if $|f'(x_0)| < 1$ and x_0 is unstable if $|f'(x_0)| > 1$.

Since each point in a two cycle is a fixed point of $f(f(x))$, we can decide the stability of 2-Cycles by looking at the derivative of $f(f(x))$.

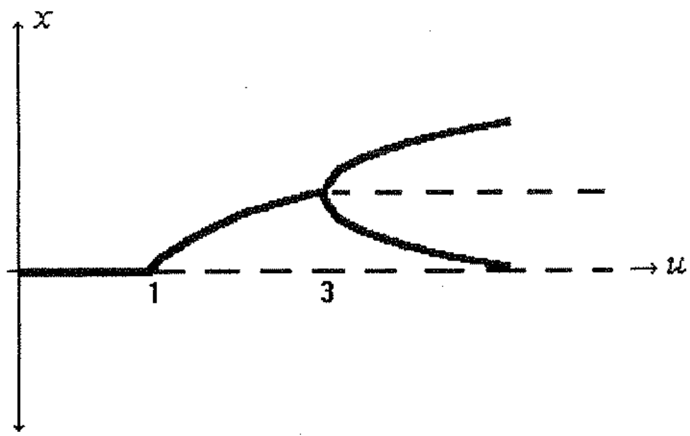
$$[f(f(x_0))]' = f'(f(x_0))f'(x_0) = f'(x_1)f'(x_0)$$

In two-dimensional systems, we use a Jacobian matrix of partial derivatives to determine stabilities.

ex: for the logistic map, $x = 0$ is stable for $\mu < 1$, then becoming unstable. For $1 < \mu < 3$, $x = \frac{\mu - 1}{\mu}$ is stable. For μ slightly greater than 3, $x = \frac{\mu + 1 \pm \sqrt{(1+\mu)(\mu-3)}}{2\mu}$ is stable, becoming unstable as μ is increased further.

2. Bifurcations

A bifurcation point is a value at which slight changes in the equation can cause major differences in the system. Often, a bifurcation brings about a change in the stability, causing a stable situation to become unstable. A minor change in a parameter of the function results in major differences in the behavior of the system. For example, look at the logistic map, $x \mapsto \mu x(1-x)$:



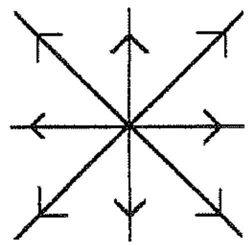
Our bifurcation plot shows the stabilities of fixed points ($x = 0$ which is stable for $\mu < 1$, $x = \frac{\mu - 1}{\mu}$ which is stable for $1 < \mu < 3$), then two-cycles ($x = \frac{\mu + 1 \pm \sqrt{(1+\mu)(\mu - 3)}}{2\mu}$ which become stable at $\mu = 3$, eventually becoming unstable later). The bifurcations that have occurred have brought us from stability to instability.

To learn more about when and how bifurcations will occur for iterated maps, we need to have tools that will give us more information about the particular family of maps in question. Eigenvalues can be a useful tool in determining the long term behavior of an iterated map. To ascertain eigenvalues for a system, we can represent the behavior of a two-dimensional family as a matrix.

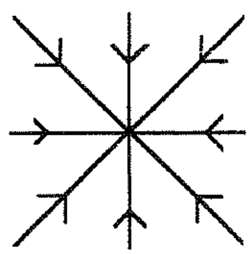
ex: $\begin{cases} u_{n+1} = \lambda_1 u_n \\ v_{n+1} = \lambda_2 v_n \end{cases}$, can be re-written as $X_{n+1} = AX_n$, where X_{n+1} and X_n are vectors of the form $X = \begin{pmatrix} u \\ v \end{pmatrix}$, and A is a two-by-two matrix, $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

The eigenvalues of the matrix, A, can tell us where our solutions will tend, based on an initial point. In this system, for example:

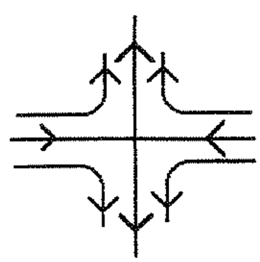
When $1 < \lambda_1 < \lambda_2$, our solutions will tend toward infinity, because the numbers always continue to grow. Our graph would look something like:



If $0 < \lambda_1 < \lambda_2 < 1$, since both of our eigenvalues are less than 1, both variables will head toward the origin, so any choice of initial point will eventually lead back to our fixed point at the origin. Our graph would be:



If $0 < \lambda_1 < 1 < \lambda_2$, then our u variable will tend toward our fixed point at the origin, while the v tends to infinity. This would result in a saddle in the graph:



For cases when $\lambda_1 < 0, \lambda_2 < 0$, or $\lambda < 0$ (both values), behavior is similar to the cases when $\lambda > 0$, except values flip back and forth across the axis, switching from positive to negative, back to positive again with each iteration. In these cases, values will still tend toward the origin or toward infinity, but the approach will be a flip-flop, or spiraling in appearance.

We summarize with a table:
(The lines denote the boundary values for the real part of λ).

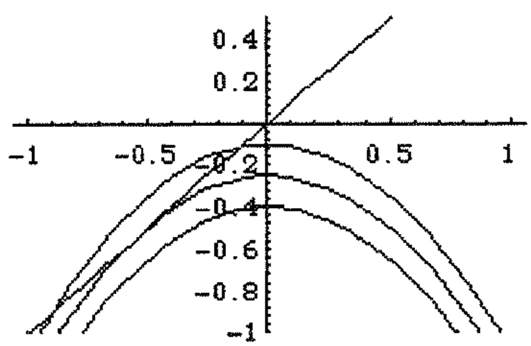
	-1	1	λ_1
λ_2	Unstable	Saddle	Unstable
1	Saddle	STABLE	Saddle
-1	Unstable	Saddle	Unstable

Within each of these regions, behavior of the map is clearly defined, but on the boundaries, when $\lambda = \pm 1$, small changes in a parameter can *greatly* change the graph. We can use bifurcation plots to explore the results that can occur at these boundary values.
Three main cases occur:

$\lambda = 1$: *saddle-node, transcritical, and pitchfork bifurcations*

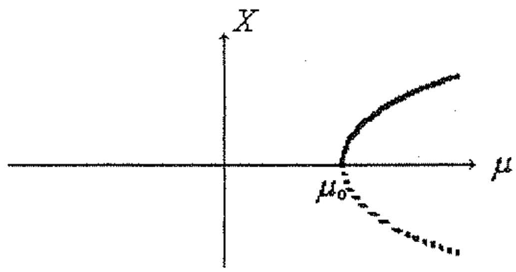
Saddle Node[3]**:**

ex: $X \mapsto \mu - X^2$, at $(X, \mu) = \left(-\frac{1}{2}, -\frac{1}{4}\right)$



Let μ_0 be the value for μ at which the bifurcation occurs. In a saddle-node bifurcation, we begin, for μ just below the bifurcation value, μ_0 , with a graph that does not cross the line $y = x$, which, in our graph, is actually the line $X_{n+1} = X_n$. So here, we have no fixed points. At the bifurcation point, μ_0 , our graph is exactly tangent to $y = x$, so we have one fixed point. Then, when $\mu > \mu_0$, our graph crosses $y = x$ twice, so we have two fixed points. This agrees with our bifurcation plot for this saddle node bifurcation example, as shown below:

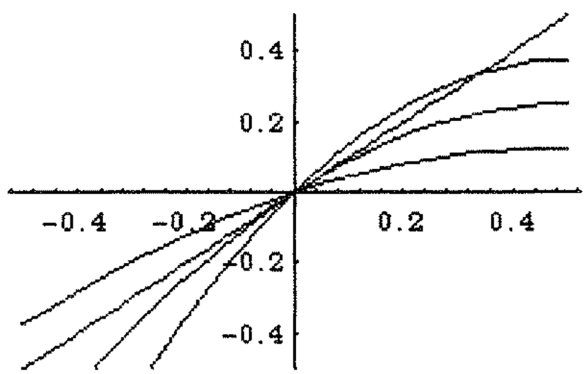
$X \mapsto \mu - X^2$:



Note that dashed lines represent unstable fixed points, while solid lines represent stable fixed points.

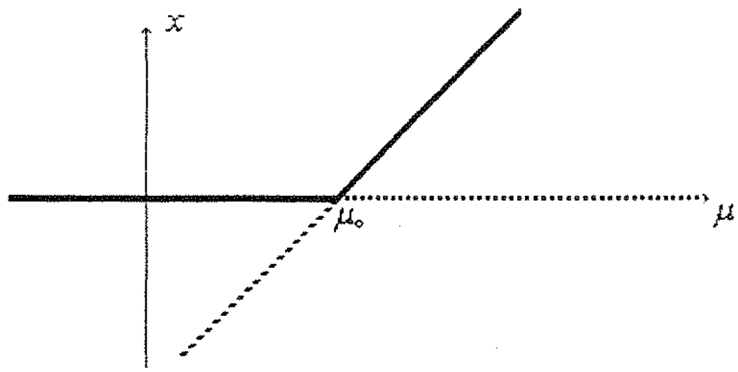
Transcritical[3]**:**

ex: $x \mapsto \mu x(1-x)$ (the logistic map), at $(x,\mu) = (0,1)$.

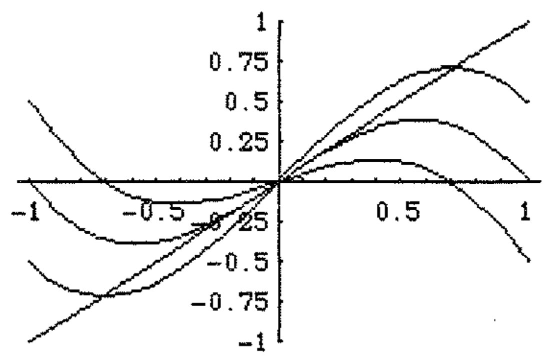


Let μ_0 be the value for μ at which the bifurcation occurs. In a transcritical bifurcation, for μ below the bifurcation value, μ_0 , the graph crosses the line $X_{n+1} = X_n$ twice. So here, we have two fixed points. At the bifurcation point, μ_0 , our graph is exactly tangent to $y = x$, so we have one fixed point. Then, when $\mu > \mu_0$, our graph crosses $y = x$ twice again, so we have two fixed points. Our corresponding bifurcation plot for this transcritical bifurcation is:

$x \mapsto \mu x(1-x)$:

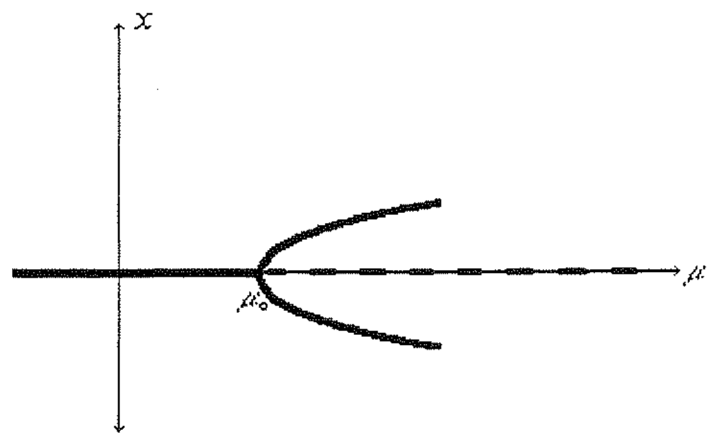


Pitchfork[3]: $(x,y) \mapsto (y, -\frac{1}{2}x + \mu y - y^3)$, at $(x,y,\mu) = \left(0,0,\frac{3}{2}\right)$.



Let μ_0 be the value for μ at which the bifurcation occurs. In a pitchfork bifurcation, we begin, for μ just below μ_0, with a graph crosses $X_{n+1} = X_n$ once. Here, we have one fixed point. At μ_0, our graph is exactly tangent to $X_{n+1} = X_n$, so we still have one fixed point. Then, when $\mu > \mu_0$, our graph crosses $X_{n+1} = X_n$ three times, so we have three fixed points. This agrees with our bifurcation plot for this saddle node bifurcation example, as shown below:

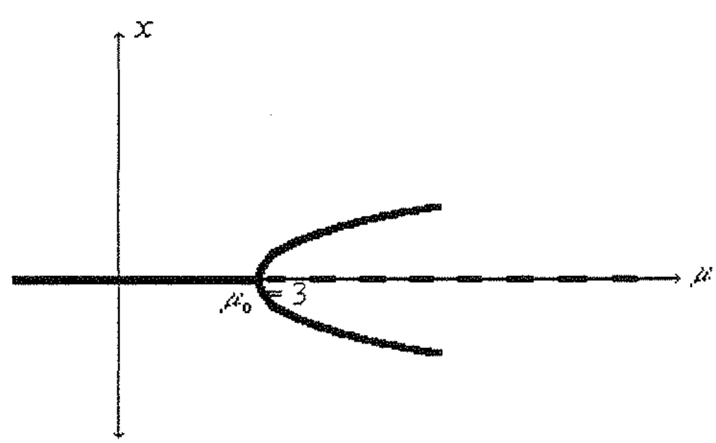
$(x, y) \mapsto (y, -\frac{1}{2}x + \mu y - y^3)$:



Note : For the pitchfork bifurcation, the pair of symmetric arcs is *not* a two-cycle. Each line is an independent fixed point, but we will be attracted to only one of them, depending on the chosen initial condition.

$\underline{\lambda = -1}$: *flip bifurcations*[3]

$(x, y) \mapsto (y, -\frac{1}{2}x + \mu y - y^3)$ undergoes a flip bifurcation at $\mu = 3$.

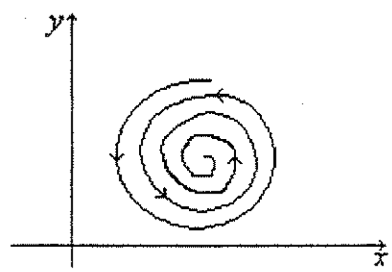


In a flip bifurcation, the two symmetric arcs which form at μ_0 are a pair of two cycles. In other words, at that value for μ, x jumps back and forth between the two arcs, instead of settling down to one fixed value. Unfortunately, the bifurcation plots do not show any difference between this flip bifurcation and a pitchfork bifurcation, in which two separate, independent fixed points occur. We must do some more detailed investigating For example, in this system, since $x \mapsto y$, and $y \mapsto -\frac{1}{2}x + \mu y - y^3$, we can let $y = x$, so $y = x = f(x,\mu) = -\frac{1}{2}x + \mu x - x^3$. In this form, we can use $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial \mu}$, and higher order partial derivatives to determine the type of bifurcation that the system undergoes at a given point.

$\underline{\lambda}$ <u>is complex modulus 1:</u> *Hopf bifurcations*[3]

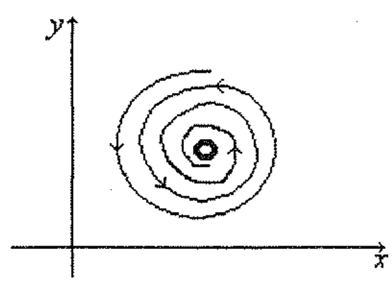
Complex eigenvalues lead to a Hopf bifurcation when the complex number is modulus 1. For complex eigenvalues, λ:

When $\underline{|\lambda| \le 1}$:



graph is swirling in (STABLE)

For $\underline{|\lambda|}$ <u>only slightly greater than 1:</u>

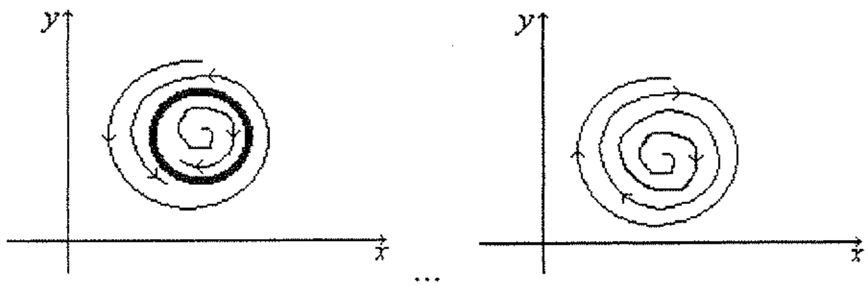


an infinitely small limit circle appears

inside which: values will swirl out

outside which: values will swirl in

$|\lambda| > 1$:

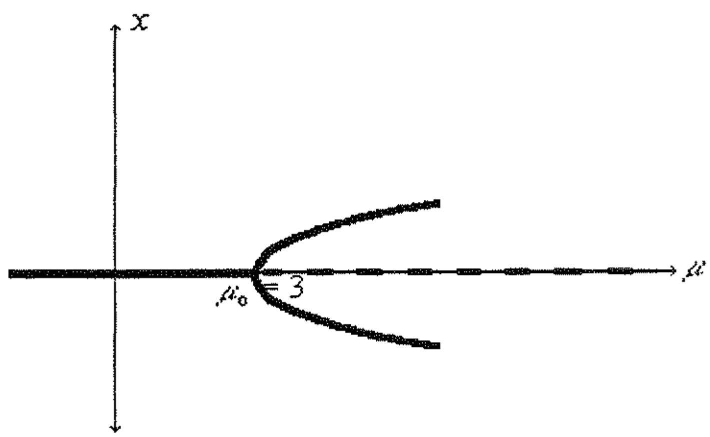


limit circle grows as values increase above 1;
then eventually, graph swirls out (UNSTABLE)

*Note that these limit circles are discrete points, not a continuous curve. The points are simply so close together that the plot appears to be continuous.

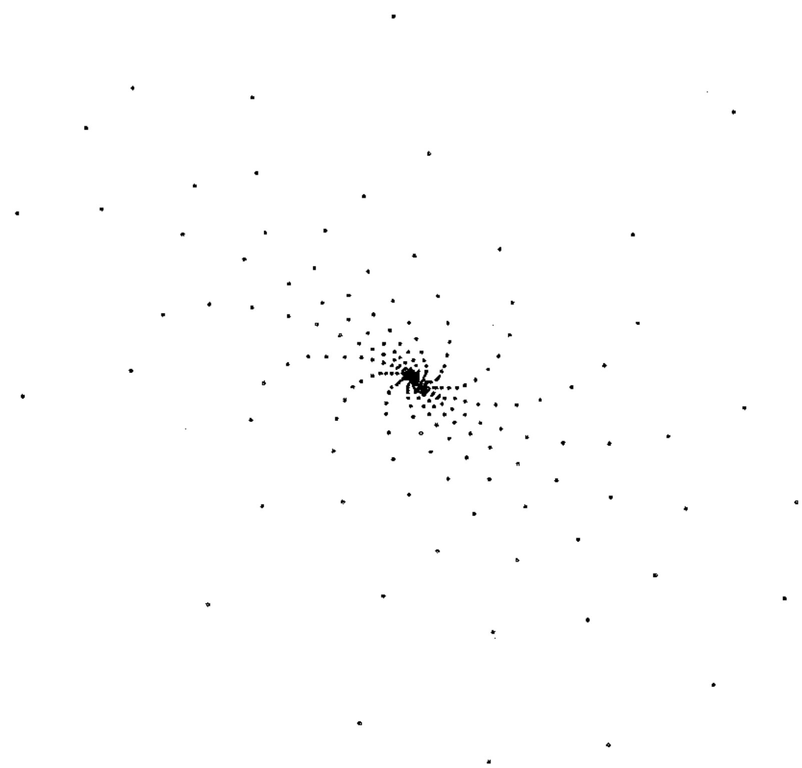
Therefore, at $|\lambda| = 1$, the Hopf bifurcation occurs - we go from a stable to an unstable situation.

In our flip bifurcation example, $(x,y) \mapsto (y, -\frac{1}{2}x + \mu y - y^3)$, at $\mu = 3$, our two cycles appear, but our stable fixed point becomes unstable. This change in the stability of the fixed point is a Hopf bifurcation at that point:

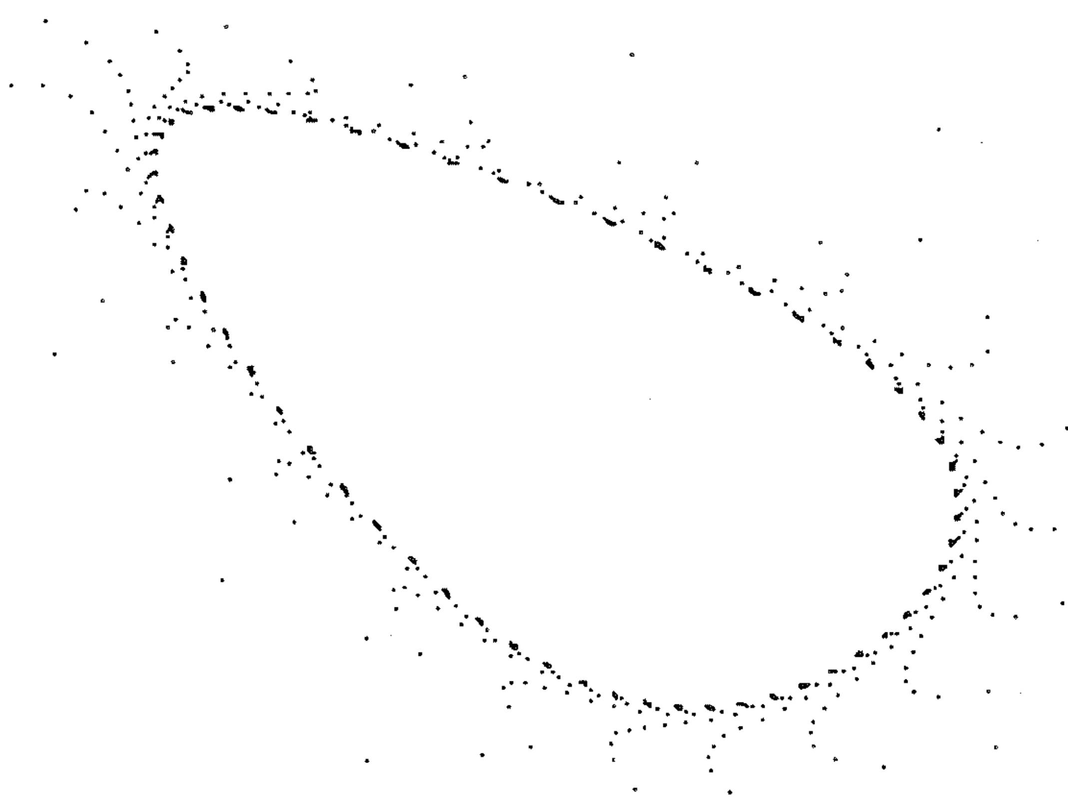


A classic example of a Hopf bifurcation occurs in the system $(x,y) \mapsto ((ax+by)e^{-0.1(x+y)})$ for values of a and b between $(a,b) = (9,15)$ and $(a,b) = (10,15)$. At (9,15), the bifurcation has not yet occurred, and for any chosen initial value, solutions swirl in. By (10,15), however, initial values chosen inside a limit circle tend to head out toward the circle, while values chosen outside the circle will head in toward the circle.[4]
As shown below:

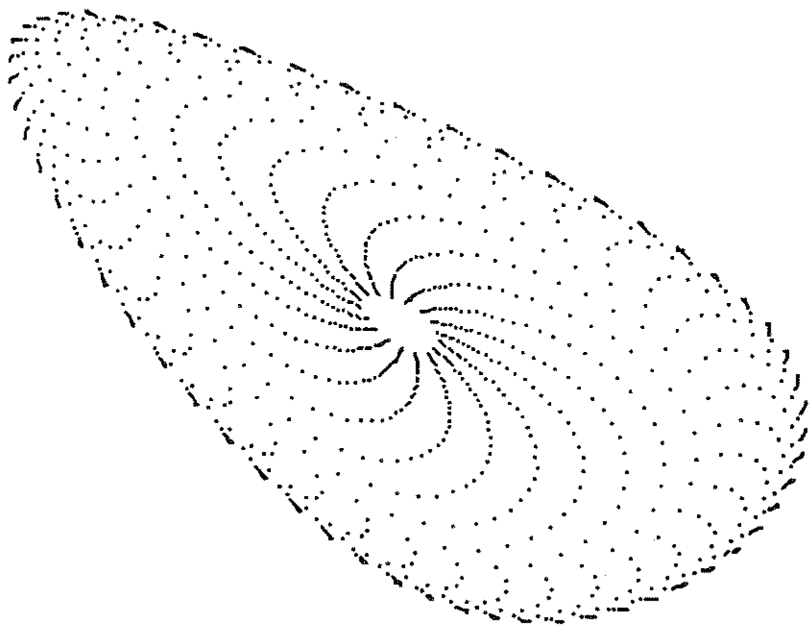
At $(a,b)=(9,15)$:



and at $(a,b)=(10,15)$:
Given an initial point *outside* of the limit cycle:



Given an initial point *inside* the limit cycle:



These plots clearly show the change in behavior of a system which occurs at a Hopf bifurcation.

3. Mimicry Map (without delay)

Mimic Map[1]:

Our original model was based upon a comparison of two insect populations. One was a population of stink bugs; the other was a population of mimic bugs that look exactly like the stink bugs but do not secrete any odors. By looking identical to the stink bugs(from here on referred to as "models") however, the mimics manage to avoid predators by fitting in with their stench-laden counterparts. To compare the population ratios of the two competing species, we used an iterated map, letting x equal the proportion of models to the total population of both mimics and models. With this original ratio, our map exists on $[0,1]$ by $[0,1]$, where $x = 0$ represents all mimics and $x = 1$ represents all models. Logically, this map must have at least one fixed point between 0 and 1, otherwise we would end up with all of one species and none of the other. We rescale x so that this fixed point occurs at $x = 0.5$.

We build the model with symmetry and rescale to $[-1,1]$ so that our iterated map is given by $x \mapsto ax^3 + (1-a)x$ on the region $[-1,1]$ by $[-1,1]$. We need $0 < a < 4$.

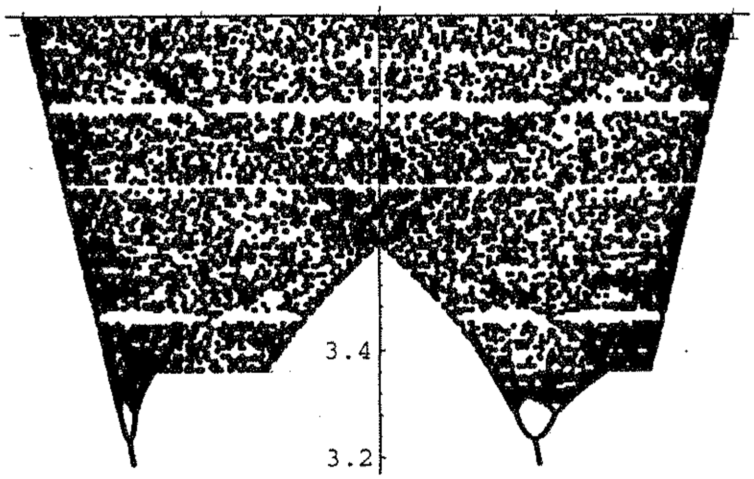
$x \mapsto ax^3 + (1-a)x$ means $x_{n+1} = ax_n^3 + (1-a)x_n$

To determine fixed points, we let $x = ax^3 + (1-a)x$, finding values where in the long run, the value of the nth iterate, x_n ,is equal to the value of the (n+1)th iterate, x_{n+1}. Solving, we find fixed points at $x = 0, \pm 1$. By looking at derivatives of $f(x) = ax^3 + (1-a)x$, we see that $x = 0$ is a stable fixed point for $0 < a < 2$, and unstable otherwise, and $x = \pm 1$ are both stable fixed points for $-1 < a < 0$. Thus we observe that a bifurcation must occur at $a = 0$, when one fixed point becomes unstable, and two others simultaneously become stable. By our description, this seems to be a pitchfork bifurcation.

To find two-cycles, we follow a similar method, this time solving for cases where $x_{n+2} = x_n$. From this, we see that a set of two-cycles, $x = \pm\sqrt{\frac{1+a}{-1+a}}$ appears at $a = 2$; thus a flip bifurcation occurs here. By continuing these calculations, we can plot further cycles and bifurcations for the mimicry map. By another method, we can calculate the long term behavior of the system for varying values of the parameter a. Using computer code, we can plot values of a for $3 < a < 4$ to get an idea of what the cycles are as a increases, without unnecessarily tedious calculations. Using Mathematica, the following program was coded[1]:

```
Bif := (
lst := {};
f[a_, x_,]:= a x^3 + (1-a) x,
a:= 3.19;
x0:= 0.1;
n:=0;
x=x0;
While[a<=4,
   While[n<=200,
     If[n>=125,
       AppendTo[lst, {N[f[a,x],10], a}]];
     x=N[f[a,x],10];n++;];
   n:=0;
   x=x0;
   a+=0.005];
ListPlot[lst])
```

The resulting graph is a bifurcation plot for the mimic model, for $3 < a < 4$:



4. Delayed Mimicry Map

Following the same idea used by Guckenheimer and Holmes for the delayed logistic map[3], a time delay was introduced into the model & mimic problem. In the mimicry map, a time delay effectively means that insects both one and two generations back affect the new generation.

Now, instead of the $x \mapsto ax^3 + (1-a)x$ iterated map without the delay, the delayed mimicry map is represented by the two dimensional system:

$$(x, y) \mapsto (y, y(ax^2 + (1-a)))$$

This another form, we see that this map is:

$$\begin{matrix} F \\ G \end{matrix} \quad \begin{cases} x_{n+1} = y_n \\ y_{n+1} = y_n(ax_n^2 + (1-a)) \end{cases}$$

Fixed points are calculated to be: $(0,0)$, $(1,1)$, and $(-1,-1)$. Simplifying the system, we see that in the long run, $x = y$ and $y = y(ax^2 + (1-a))$, so our function is $f(x, y) = y(ax^2 + (1-a))$. Using this and its partial derivatives (and the Jacobian matrix, $\frac{\partial(F,G)}{\partial(x,y)}$ of the two equation system), we find that all three fixed points are stable for $0 < a < 1$; (0,0) stays stable until $a = 2$.

For values of a less than 2, any initial value except those at the corners of the square $[-1,1]\text{x}[-1,1]$ will be attracted into the origin

Proof:
We use the form $x_{n+1} = x_n(ax_{n-1}^2 + (1-a))$
So:

$$|x_{n+1}| = |x_n||ax_{n-1}^2 + (1-a)|$$

We know that our initial point (x_n, x_{n-1}) is within the region $-1 < x < 1$, $-1 < y < 1$, so $|x_n| \le 1$ and $|x_{n-1}| \le 1 \ \forall n$.
$|x_{n-1}| < 1 \ \Rightarrow \ |ax_{n-1}^2 + (1-a)| < 1$ for $0 < a < 2$, because on this region, $(ax^2 + (1-a))$ is a series of parabolas, with vertex on the line $x = 0$, and maximum endpoints at $(1,1)$ and $(-1,-1)$.
And if $|ax_{n-1}^2 + (1-a)| < 1$, this implies that: $|x_{n+1}| < |x_n| < 1$.
So as $n \to \infty$, $x_n \to c$ where c is a constant such that $|c| < 1$.

In our expression, $|x_{n+1}| = |x_n||ax_{n-1}^2 + (1-a)|$, $|ax_{n-1}^2 + (1-a)| < 1$ if $0 < a < 2$ and $|x_{n-1}| < 1$. However, if $x_{n-1} = \pm 1$, then $x_{n+1} = x_n$. But on the next iteration, $x_{n+2} = x_{n+1}(ax_n^2 + (1-a))$, $x_{n+2} \to 0$ unless x_n also equals ± 1. The corner points, $(x_n, x_{n-1}) = (\pm 1, \pm 1)$, will not approach zero. Instead, they go to the fixed points at $(1,1)$, and $(-1,-1)$.

Now, we must prove that $c = 0$ (*i.e.* that x, y approach zero, not a constant). To prove this, note that
$x_{n+1} = x_n(ax_{n-1}^2 + (1-a)) \quad \Rightarrow \quad \frac{x_{n+1}}{x_n} = ax_{n-1}^2 + (1-a)$
Assume c is a constant, $c \ne 0$, and $|c| < 1$. Then as $n \to \infty$, $x_{n+1} \to c$, and $x_n \to c$. As $x_{n+1}, x_n \to$ a value (c) asymptotically, x_{n+1} and x_n become closer and closer to each other in value. $\frac{x_{n+1}}{x_n} \to \frac{c}{c} \to 1$.

Overall, as $n \to \infty$, $\frac{x_{n+1}}{x_n} \to 1$.

$$\frac{c}{c} = 1 = \frac{x_{n+1}}{x_n} = ax_{n-1}^2 + (1-a)$$
$$1 = ax_{n-1}^2 + (1-a)$$
$$a = ax_{n-1}^2$$
$$1 = x_{n-1}^2$$
$$x_{n-1} = \pm 1$$

So as $n \to \infty$, $x_{n+1} \to c$

$x_n \to c$

and $x_{n-1} \to \pm 1$.

But this would mean that $c = \pm 1$, which contradicts our basic assumption that $|c| < 1$.

Also, we note that on the boundaries (when $x_n = \pm 1$), because $x_{n+2} = x_{n+1}$, we land on the line $y = x$.

$x_{n+1} = x_n(ax_{n-1}^2 + (1-a))$

When $x_n = 0$, $x_{n+1} \to 0$ immediately. Also, if $ax_{n-1}^2 + (1-a) = 0$ (*i.e.* if $x_{n-1} = \pm\sqrt{\frac{a-1}{a}}$, then $x_{n+1} \to 0$ immediately.

But if $x_n \neq 0$ and $ax_{n-1}^2 + (1-a) \neq 0$, then how can x ever reach exactly 0? We may not necessarily land exactly on the lines $x_n = 0$ or $x_{n-1} = \pm\sqrt{\frac{a-1}{a}}$, on any iteration, so we may never jump directly in to zero. In such cases, we must approach zero asymptotically instead. OR, at least approach some other constant, c, asymptotically ($|c| < 1$), because we know that the value for x decreases with each iteration.

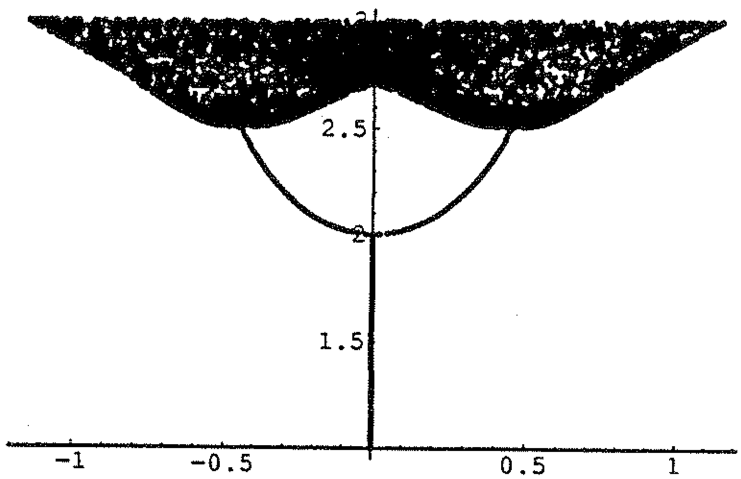
Overall, this delayed mimicry map displays a wide array of strange and inexplicable phenomena. The behavior of second iterates, the development of a strange attractor as our parameter, a increased, and the sudden and clear cycles in the map's behavior are only a few.

Two cycles occur at $\left(\sqrt{\frac{a-2}{a}}, -\sqrt{\frac{a-2}{a}}\right)$ and $\left(-\sqrt{\frac{a-2}{a}}, \sqrt{\frac{a-2}{a}}\right)$, becoming stable at $a = 2$. So we know that there is a flip bifurcation at $a = 2$. Checking with the eigenvalues of our Jacobian matrix, we see, indeed, that at $a = 2$, $\lambda = 1 - 2 = 1$, so we can be certain that this is a flip bifurcation.

The Jacobian also shows that in this system, complex eigenvalues can occur. Upon closer inspection, we find that at $a = 2.5$, $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ (i.e. λ is complex modulus 1). At this point, therefore, a Hopf bifurcation occurs. (Note that $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ is the cube root of unity. The iteration plot for this graph is a spiral with three parts. In the delayed logistic map's plot[3], the spiral had 6 parts, and indeed, the eigenvalues at that point ($\mu = 2$)

were $\lambda = e^{\pm i\pi/3}$, or sixth roots of unity. This leaves question as to whether there may be a relation between roots of unity and the formation of the iterated map, but no conclusion could be drawn as of yet.)

To verify these first fixed points and bifurcations, we refer to a bifurcation plot of the delayed mimicry map, coded in Mathematica:



From this general, overall plot, we see the fixed point for $a < 2$; then the flip bifurcation at $a = 2$; and the darkened, filled-in lines of a Hopf bifurcation at $a = 2.5$.
(Note that for this two dimensional system, to create a function for our program to plot, we must solve our system for x alone. We know that

$$x_{n+1} = y_n \Rightarrow y_{n+1} = x_{n+2}$$

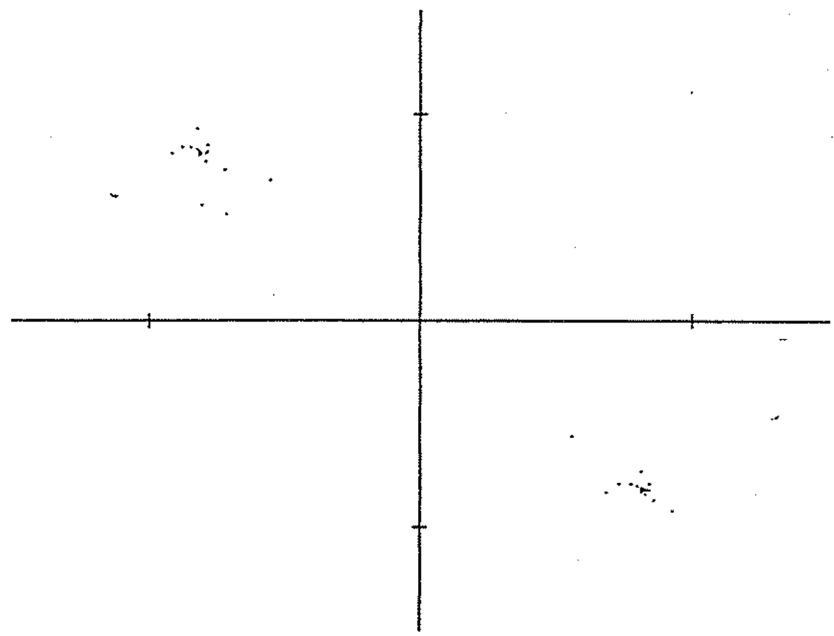
and so

$$y_{n+1} = y_n(ax_n^2 + (1-a)) \Rightarrow x_{n+2} = x_{n+1}(ax_n^2 + (1-a))$$

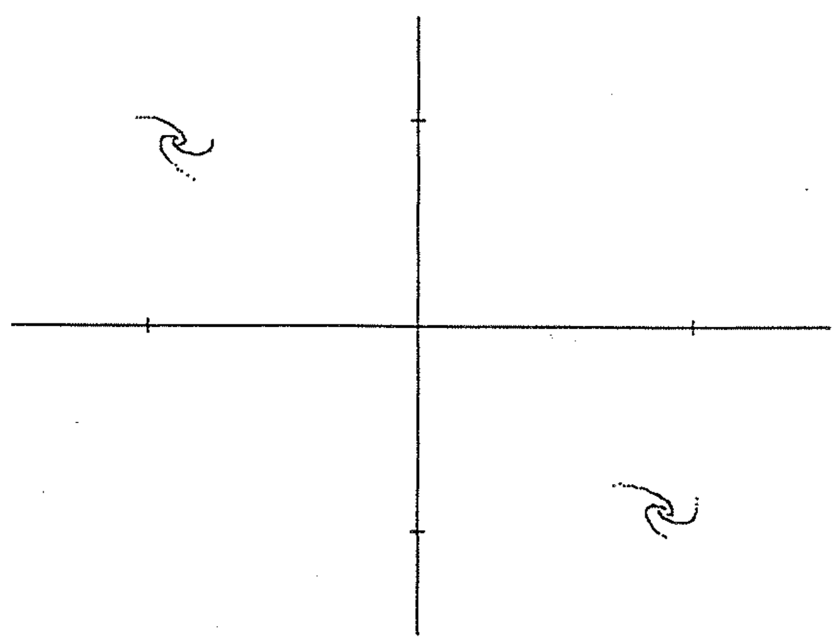
Thus we can use $f = x_{n+2} = x_{n+1}(ax_n^2 + (1-a))$ to calculate long term values in our program.)

Looking at our iteration plots, we can see the occurrence of the flip bifurcation.

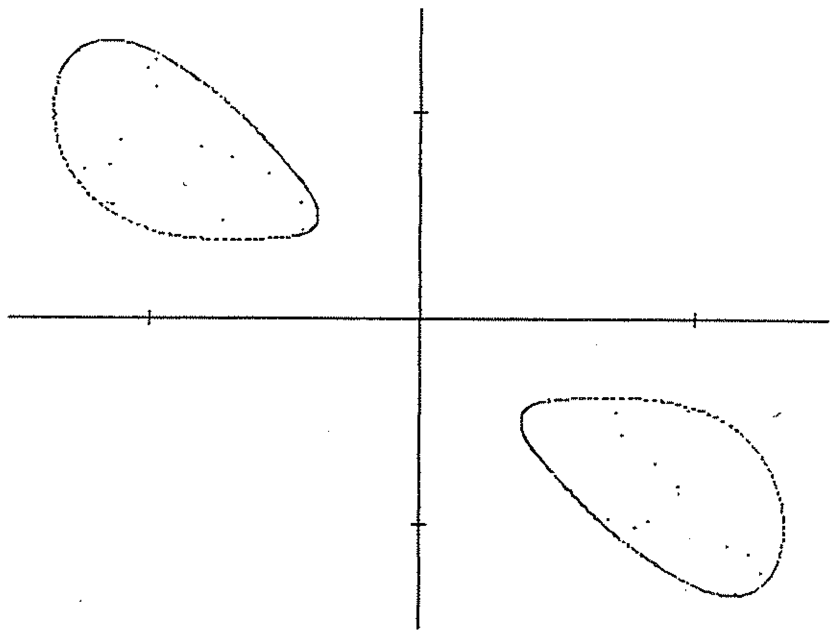
At $a = 2.4$, any initial point in the region $[-1,1]$ by $[-1,1]$ sends us spiraling in toward to symmetric points in the second and fourth quadrants:



At $a = 2.5$, our spirals are attracted to, but never quite reach an infinitely small limit circle in their centers:

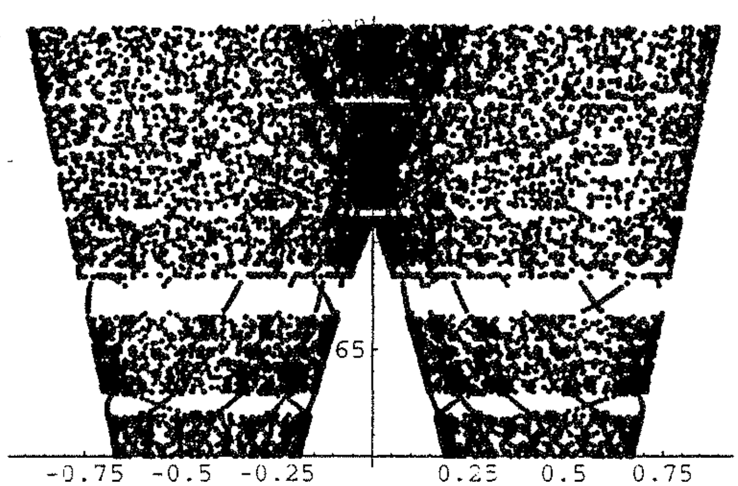


At $a = 2.6$, clear limit circles can be seen, and if we begin with an initial point inside of one of them, iterations will send points out to the circle. Likewise, points chosen outside of the limit circle will also approach it:

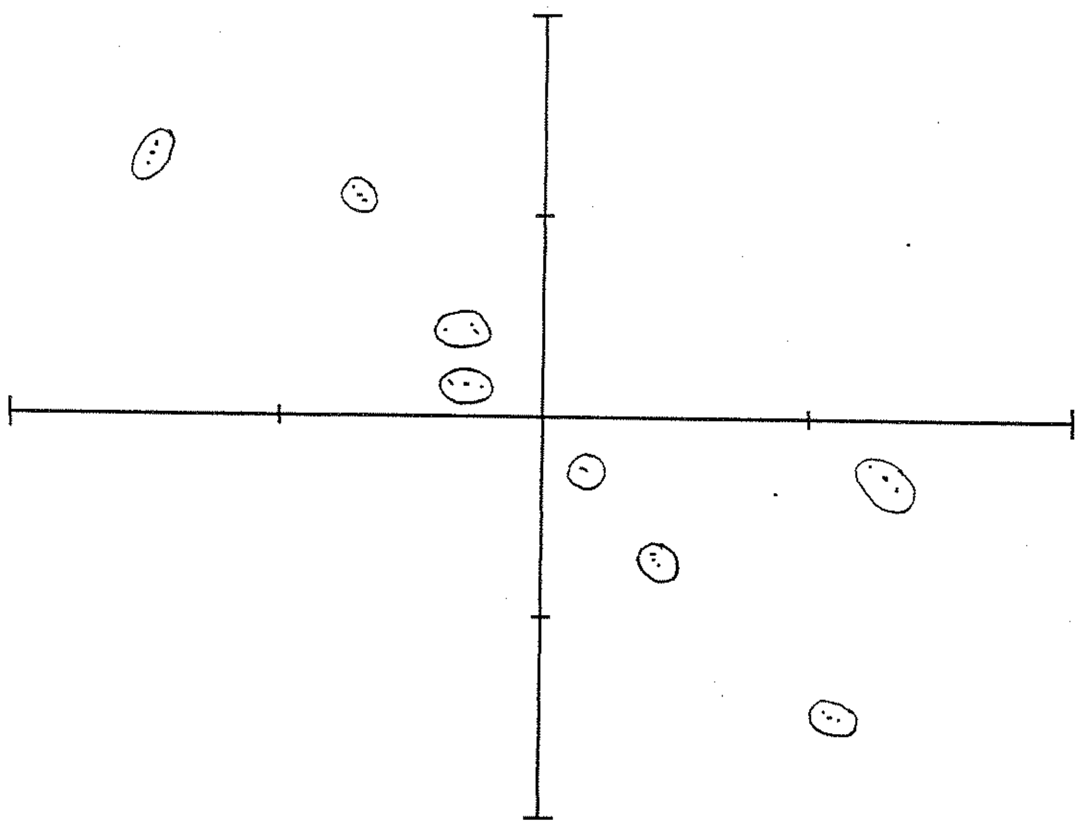


Many other interesting phenomena can be observed for this complex and dynamic system.

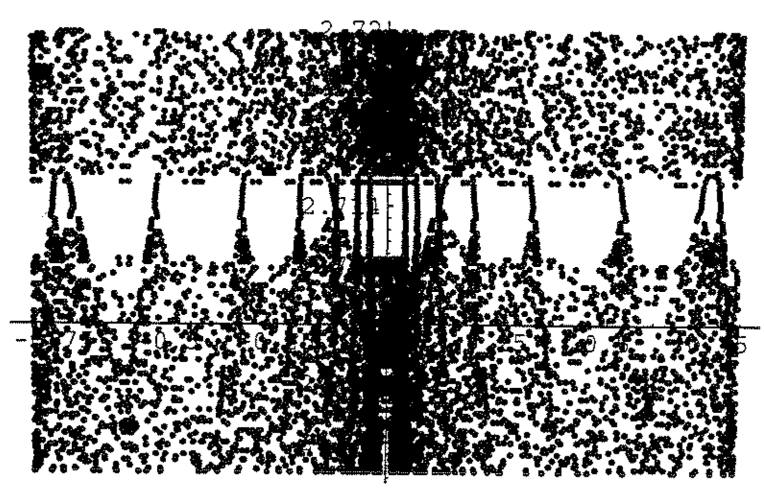
Looking at a more detailed bifurcation plot, for $2.6 \le a \le 2.8$, we see that a stable 8-cycle appears at $2.67 - 2.68$:



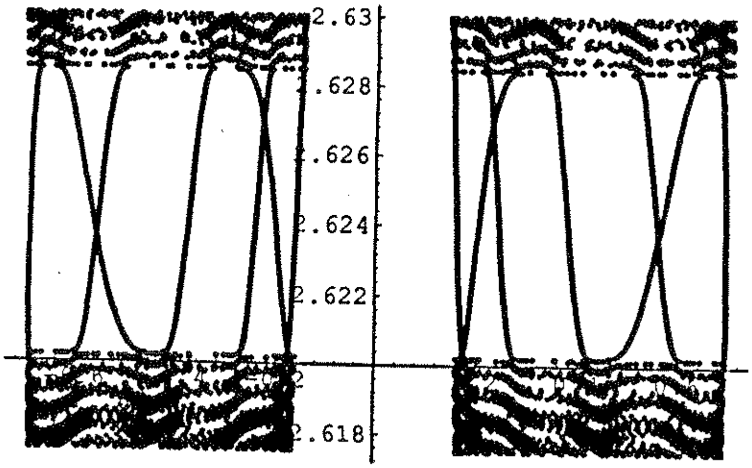
Comparing to an iteration plot at $a = 2.68$, we find that any initial guess sends us quickly to a set of eight points, through which is continues to jump forever, thus proving the existence of our eight-cycle:

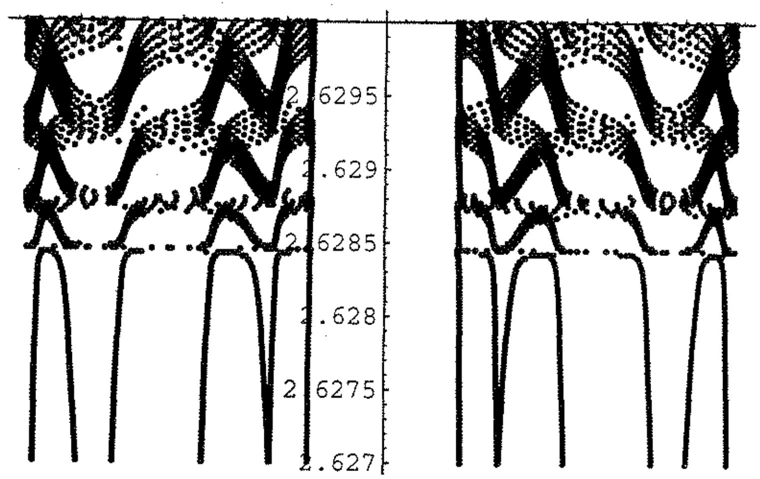


An even closer examination of the bifurcation plot reveals a similar stable 16-cycle at $a \approx 2.714$, which can be supported by iteration plots, as well:



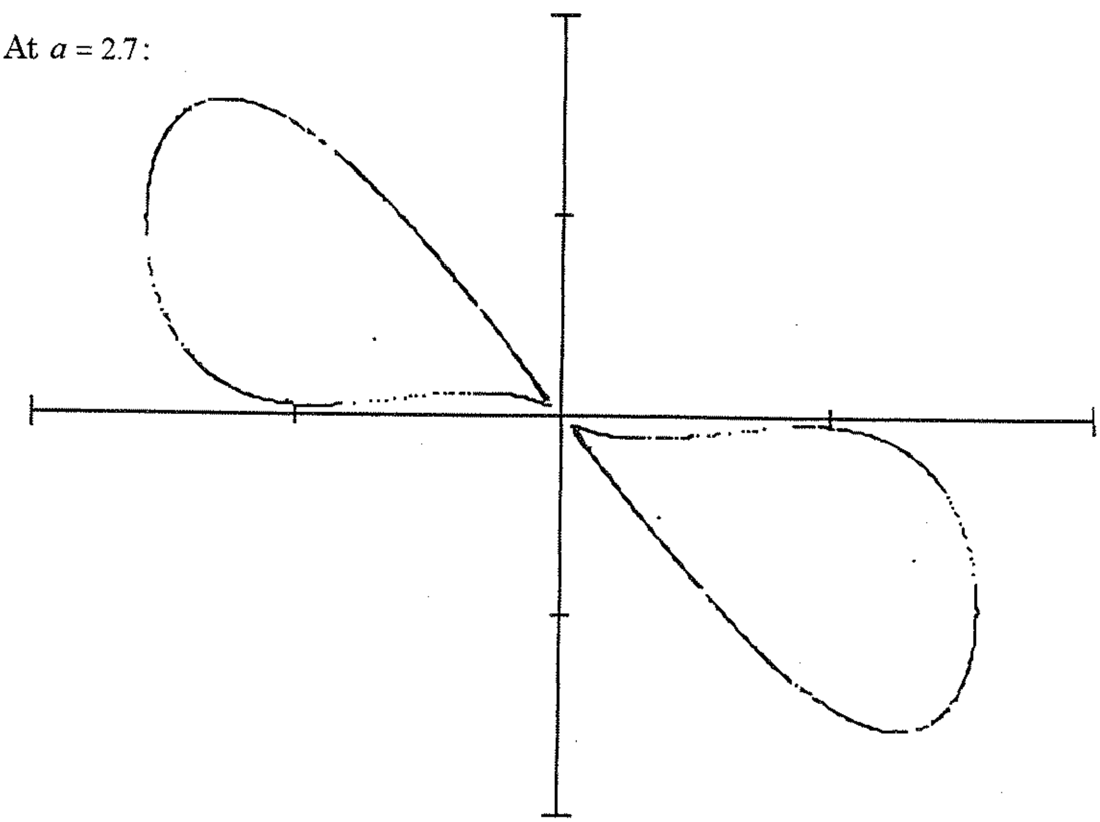
Detailed bifurcation plots also reveal an amazing amount of symmetry and pattern formation in this plot. The following detailed bifurcation plots show the precise detail in this system:



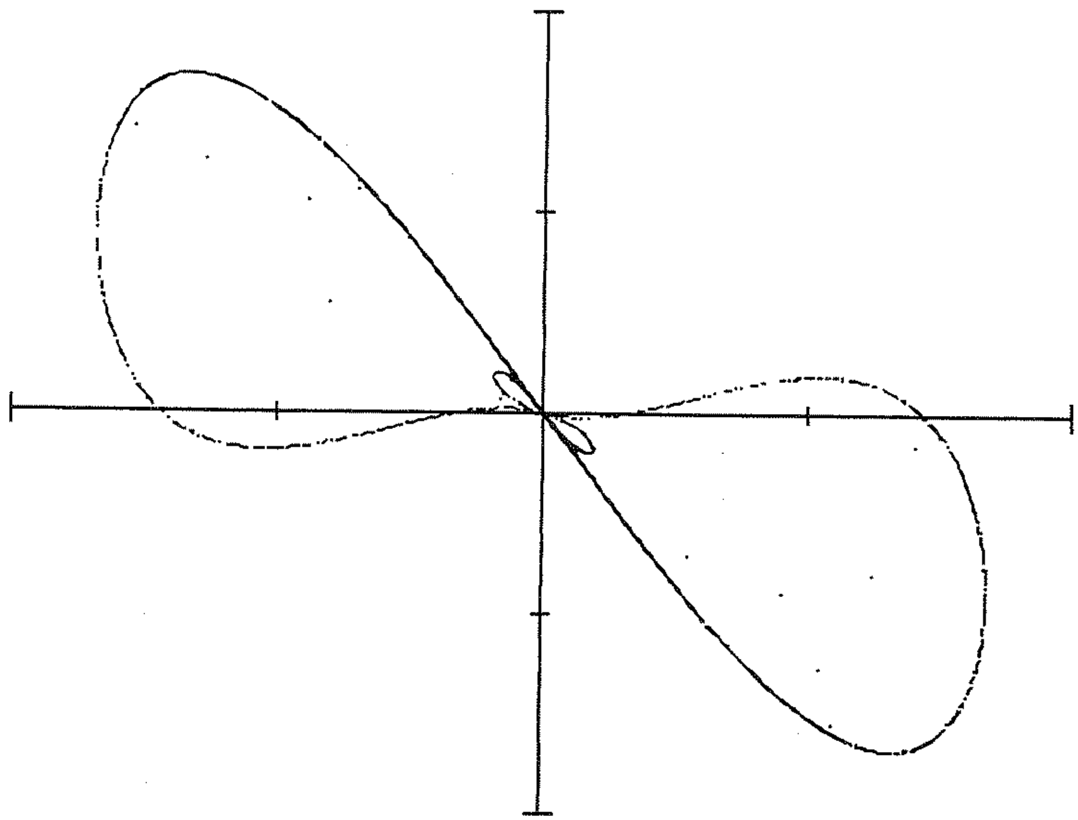


Analysis of the iteration plots shows fascinating intricacy, as well. As a increases, our loops grow and eventually join, forming homoclinic loops at (0,0).

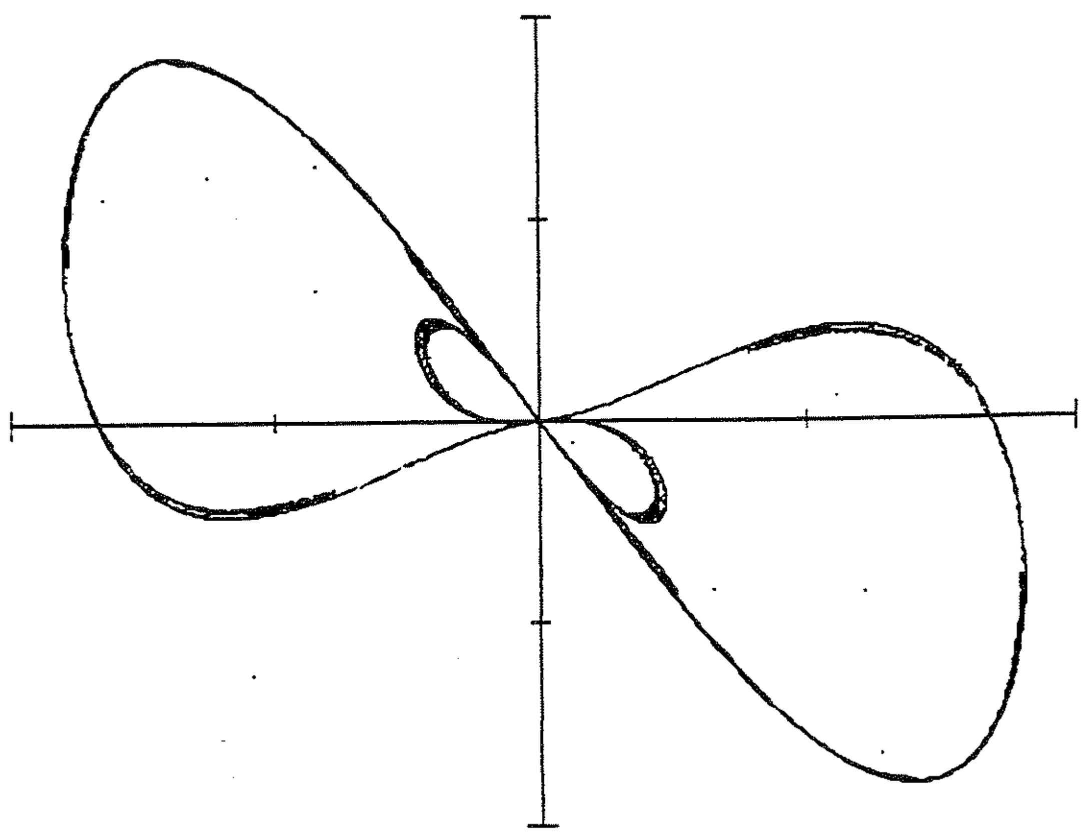
At $a = 2.7$:



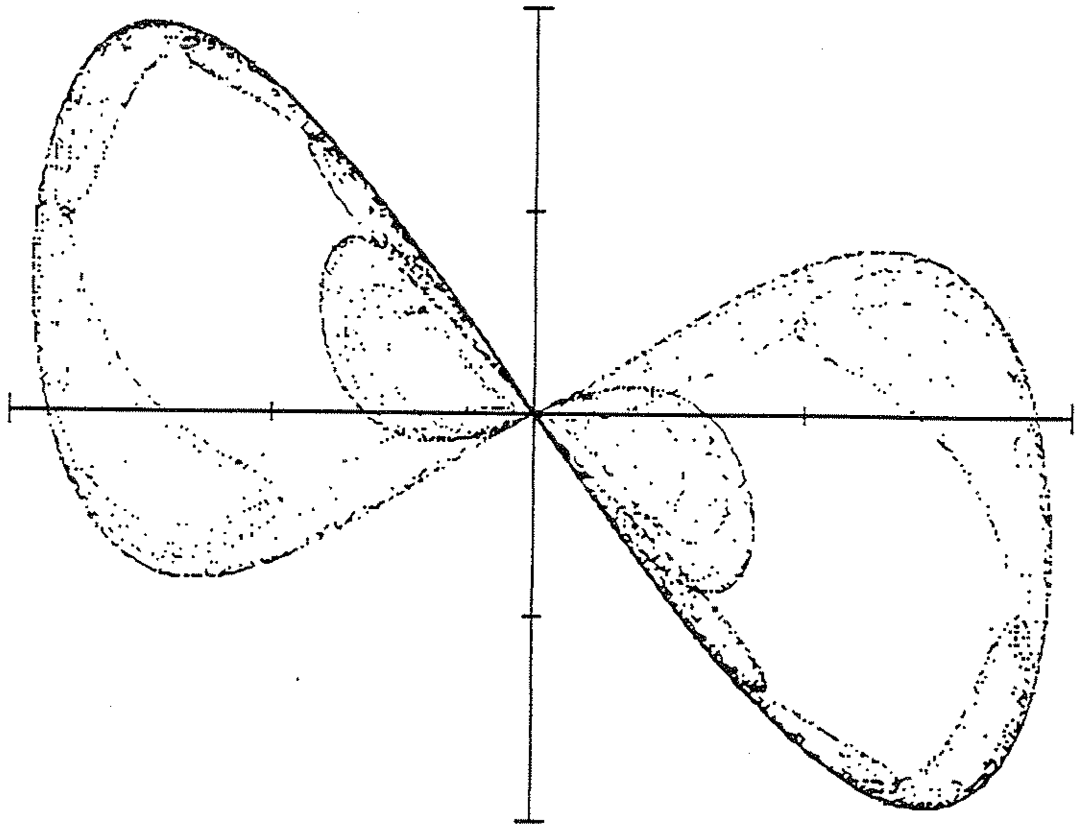
At $a = 2.75$ a new inner loop forms:



At $a = 2.8$, our pattern of iterations has become even more complex:



By $a = 2.85$, an extremely involved strange attractor has formed:



Beyond this value, our attractor continues to develop, but values begin to leave the region $[-1,1]$ by $[-1,1]$, making solutions which for our mimicry map are impossible in the physical world. By $a = 3$, though, values head off to infinity, no matter what the initial guess, and our entire system becomes unstable.

Endnotes

1. Wimmer, Barry; Mathematical Model for Mimicry; M.S. thesis -1997
2. May, Robert M.; "Simple Mathematical Models With Very Complicated Dynamics"; Nature; June 10, 1976
3. Guckenheimer, John & Holmes, Philip; Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields; 1983; Springer-Verlag; pp159-164
4. Oster, G.; "The Dynamics of Non Linear Models With Age Structure"; MAA Studies in Mathematical Biology Part II: Populations and Communities; Levin, S.A.(editor); 1978