Stalling's Lemma and Diagrammatic Reducibility

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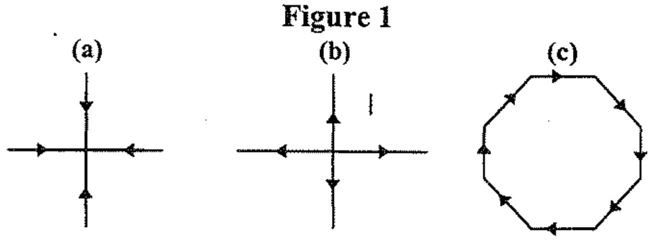
Introduction

Stallings provides in "A Graph-Theoretic Lemma and Group-Embeddings" [5] a simple lemma about directed graphs on the sphere and applies it to prove that certain classes of equations over groups are solvable. Prischepov develops in "Stallings' Lemma and Equations Over Groups" [4] an interesting variation on Stallings' method. In this paper we will present a generalization of Prishchepov's technique and apply it to prove that certain classes of presentations are aspherical.

Section 1: Stallings' Lemma

Stallings' Lemma states that given a finite oriented graph Γ, that is non-empty and without isolated vertices, embedded in the 2-sphere S^2, then there will exist at least two consistent items within Γ. The term graph as used above denotes what is called a multigraph. In group theory a graph consists of finitely many vertices and edges where multiple edges and loops are allowed. A consistent item is either a sink or a source or a consistently oriented region of the graph on S^2. A sink (source) is a vertex such that all adjacent edges are oriented towards (away from) the vertex respectively. A consistently oriented region is a simply connected region of S^2-Γ such that all edges of the boundary of the region are oriented the same way, i.e. clockwise or counterclockwise. Note that without loss of generality we can assume that Γ is a connected graph. Stallings shows this to be true by relying upon the fact that if Γ has n connected components then we can create a graph which is connected by adding n-1 edges to Γ. These additional edges do not add any new sink or source vertices and the regions containing these new boundaries cannot be consistently oriented regions since each of these new edges must be traversed twice, once in each direction on the boundary of such a region. I shall now prove Stallings' Lemma for a connected graph utilizing the weight test.

Figure 1



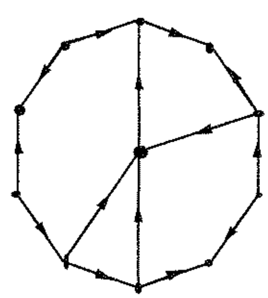
Proof: In the following, "corner" will denote a corner of a region of the graph, i.e. a vertex together with its two adjacent edges in the boundary of that region. (Think of the region as a room with the edges as its walls and of a corner of this room.) A corner is called a sink (source) corner if both edges of the corner are oriented towards (away from) the vertex. We will assign weights to corners. (You can think of a weight as an angle.) Assign each corner that is neither a sink nor a source a weight of π and each corner that is a sink or a source a weight of 0. Define the angle sum of a vertex v_i to be the sum of the corner weights adjacent to v_i and denote it by $a(v_i)$. Further define the angle sum of region R_i to be the sum of the corner weights within R_i and denote it by $a(R_i)$. This implies that a sink or source vertex must have an angle sum of 0, and that a consistently oriented region R_i must have an angle sum of nπ. The total curvature of a surface is defined to be the summation of its local curvatures. The local curvature of a vertex or a face is defined to be $2\pi - a(v_i)$ or $a(R_i)-(n-2)\pi$ respectively. There exists a theorem known as the weight test which guarantees that for any assignment of weights to the corners of a connected graph embedded on S^2 the sum of all the local curvatures will be 4π (see for example [1] or [3]). Thus providing us with the following equation: $4\pi = \sum_{i=1}^{n}(2\pi - a(v_i)) + \sum_{i=1}^{m}(a(R_i)-(n-2)\pi)$. If Γ is embedded in S^2 and is connected then there exists at least one vertex whose angle sum is less than 2π or there exists at least one region whose angle sum is greater than (n-2)π. Assume that there exists a vertex v_i with angle sum less than 2π that is not a sink or a source. This implies that there exists a corner of v_i having a weight of π and that all other corners have a weight of 0. Let $c_1, c_2, \ldots, c_n$ be the corners of v_i, and let e_i, e_{i+1} be the edges of c_i. Let c_i have weight π. This implies that if e_i is oriented towards v_i then e_{i+1} must be oriented away from v_i. Since c_{i+1} has weight 0, it follows that e_{i+2} must be oriented away from v_i. Similarly it can be shown that all e_j, $j \neq i$, are oriented away from v_i. Since c_{i-1} has weight 0, and e_{i-1} is oriented away from v_i, it follows that e_i must be oriented away from v_i, a contradiction. A similar contradiction will be arrived at if we first assume that e_i is oriented away from v_i. Therefore a vertex that is neither a sink nor a source must have a weight greater than or equal to 2π. Now assume that there exists a region R_i which is not consistently oriented whose angle sum is greater than (n-2)π. This implies that there are n-1 corners with weight π and one corner with weight 0. Let $c_1, c_2, \ldots, c_n$ be the corners of R_i, and let e_i, e_{i+1} be the edges of c_i. Let c_i have weight 0. This implies that if e_i is oriented clockwise then e_{i+1} must be oriented counter-clockwise. Since c_{i+1} has weight π it follows that e_{i+2} must be oriented counter-clockwise. similarly it can be shown that e_j, $j \neq i$, must be oriented counter-clockwise. Since c_{i-1} has weight π and e_{i-1} has counter-clockwise orientation then e_i must have counter-clockwise orientation, a contradiction. A similar contradiction will be achieved if e_i is first assumed to be oriented counter-clockwise. Therefore a region which is not consistently oriented has angle sum less than or equal to (n-2)π. Since any vertex or face which is not a consistent item of the graph has curvature ≤ 0, and the consistent items each have curvature 2π, it follows that there exist at least two consistent items in Γ.

Section 2: Prishchepov's Technique

Before describing Prischepov's Technique it is important to have an understanding of an equation over a group. A polynomial over a group G in an indeterminate X is defined to be an element of the group G[X]=G*X.. This, as Howie notes in [2], "allows us to consider a polynomial as a word $p(x) \equiv g_0 X^{\lambda(1)} g_1 \ldots X^{\lambda(k)} g_k$ with each g_i a nontrivial element of G and each $\lambda(i)$ a non-zero integer." (The elements g_i are the coefficients.) An equation in X over G is of the form $p(x) = q(x)$ where p(x) and q(x) are both elements of G[X]. Without loss of generality we can assume q(x) to be 1. (By rewriting the equation as $p(x)(q(x))^{-1} = 1$ and renaming the left side p(x).)

Prishchepov used Stallings' Lemma to exhibit a new class of equations that have a solution over any torsion-free group. In his proof that such equations were solvable Prishchepov developed a method of constructing an ancillary graph associated with a diagram of equations which exhibited certain properties. The condition of relevance to our discussion is that each equation contain sub-words $tb_j^i t$ and $t^{-1} b_k^i t^{-1}$.

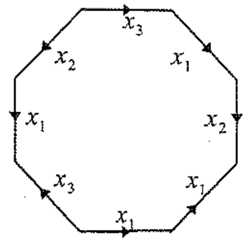
Figure 2



Section 3: Extension of Prishchepov's Technique.

We shall apply Prishchepov's technique of constructing an ancillary graph on a diagram of equations over groups, to a technique of constructing an ancillary graph on a diagram over a presentation and generalize Prishchepov's result at the same time. Note that a "presentation" of a group G means a finite presentation of G by generators and defining relations. A spherical diagram over a presentation is a tessellation of the sphere whose oriented edges and oriented faces are labeled by the generators and the relations, or their inverses, respectively. More precisely, let a relation R_i of a presentation $P = \langle x_1, x_2, \ldots, x_n | R_1, R_2, \ldots, R_m \rangle$ be represented by a polygon whose edges are directed and labeled by generators such that the edges of the polygon read the word R_i up to cyclic permutation and inversion (figure 3).

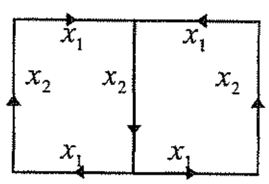
Figure 3



$$R_i = x_1^2 x_2^{-1} x_1^{-1} x_3^{-1} x_2 x_1 x_2^{-1}$$

For convention we shall read counter-clockwise around the polygon. A *diagram* is then formed by gluing copies of the polygons corresponding to the relations and their inverses together along edges so that they form a surface. These polygons are said to be the "2-cells" of the diagram. The edges of the polygons can only be glued together if their labels and directions match. In the gluing process it is possible to use a polygon corresponding to a relation or its inverse multiple times. In the following sections we will only consider *spherical diagrams*, i.e. diagrams where the surface is a sphere. A diagram is said to be *reduced* if it contains no elementary folds. An elementary fold occurs when two relations glued together are mirror images of each other across their common edge. A presentation is said to be *diagrammatically reducible* if every spherical diagram associated with it contains an elementary fold (figure 4).

Figure 4



Now assume that P is a presentation such that each relation contains at least one pair of consecutive edges oriented clockwise and a least one pair of consecutive edges oriented counterclockwise. This implies that the cyclic word R_i must contain consecutive letters $x_{i_j}, x_{i_{j+1}}$ and consecutive letters $x_{i_k}^{-1}, x_{i_{k+1}}^{-1}$. Further assume that each relation in the presentation is cyclically reduced. This means that no cyclic permutation of a relation contains sub-words $x_i x_i^{-1}$ or $x_i^{-1} x_i$. Let D be a spherical diagram over P. Construct an ancillary graph Γ in the following fashion. At the center of each face of the diagram place a vertex v_i and call it an *interior vertex* (of the ancillary graph). Along each boundary of a 2-cell of the diagram locate each vertex that lies between a pair of consecutive edges that are oriented the same way (as described above) and call them *boundary vertices* (of the ancillary graph). Next connect the interior vertex of each face to all the boundary vertices of the corresponding face by a directed ancillary line according to the following convention: if the boundary vertex corresponds to the consecutive edges $x_{i_j}, x_{i_{j+1}}$ then the ancillary line should begin at the interior vertex and end at the boundary vertex, and begin at the boundary vertex and end at the interior vertex otherwise (figure 2). Note that Γ is a sub-graph of the dual of the graph consisting of the edges and vertices of the diagram.

Section 4: Test for Diagrammatic Reducibility

Having established a method of constructing an ancillary graph Γ associated with a diagram over a presentation we now show how this technique may be used to demonstrate that a presentation is diagrammatically reducible. In the following we will consider presentations that satisfy the condition (p):

each relation R_i of P is cyclically reduced, has at least 2 consecutive edges oriented clockwise, and at least 2 consecutive edges oriented counter-clockwise.

Given such a presentation and a spherical diagram over this presentation construct an ancillary graph as instructed in section 3.

Lemma: Under the above assumptions, the ancillary graph Γ will contain at least two consistently oriented regions.

Proof: Assume the ancillary graph does not have two consistently oriented regions, then by Stallings' Lemma there must exist a sink or a source. Now assume that there exists a midpoint of a 2-cell that is a sink. This implies that all ancillary lines adjacent to this vertex must be oriented towards the center. Then all consecutive edges oriented identically of the 2-cell must be oriented counter-clockwise. However, the presentation requires that each 2-cell has at least one pair of consecutive edges oriented clockwise and at least one pair of consecutive edges oriented counter-clockwise. Thus, the midpoint of a 2-cell cannot be a sink of the ancillary graph. Similarly, a midpoint cannot be a source of the ancillary graph. Then there must exist a vertex **v** along the boundary of some 2-cells which is a sink or a source of the ancillary graph. Assume v is a sink of the ancillary graph, this implies that each 2-cell contributing to the ancillary graph at v must have clockwise orientation of its two consecutive edges that meet at v. Label the edges around v by $\{e_1,...,e_n\}$. Let e_1 and e_2 be two consecutive edges having such an orientation. Now if e_2 and e_3 do not have counter-clockwise orientation, then they must form a source corner. By induction e_i and e_{i+1} must form a source corner. This implies that e_n is oriented away from v. Since e_1 is oriented towards v, then e_n and e_1 have counter-clockwise orientation. Thus v cannot be a sink of the ancillary graph. Similarly, v cannot be a source of the ancillary graph. Therefore the ancillary graph must have two consistently oriented regions.

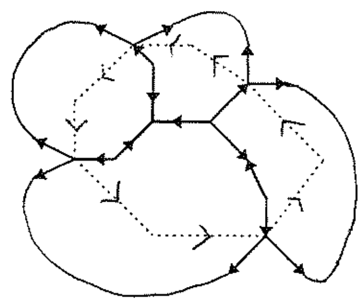
Thus if we can show that for a certain presentation P the ancillary graph of any reduced spherical diagram over P cannot contain a consistently oriented region, then this contradicts Lemma 1 and hence shows that there are no reduced spherical diagrams over P, i.e. the presentation is diagrammatically reducible. Before continuing, some terminology must be defined. A segment of a relation beginning at the vertex of a pair of consecutively oriented edges and ending at the vertex of the next pair of consecutively oriented edges will be called a *positive segment* (p_i) if one of these pairs of consecutively oriented edges is oriented clockwise and the other pair is oriented counter-clockwise such that both pairs are oriented away from the center of the segment (figure 5a). Alternatively, a segment of a relation beginning at the vertex of a pair of consecutively oriented edges and ending at the vertex of the next pair of consecutively oriented edges will be called a *negative segment* (n_i) if one of these pairs of consecutively oriented edges is oriented clockwise and the other pair is oriented counter-clockwise such that both pairs are oriented towards the center of the segment (figure 5b).

Figure 5

(a) **(b)**

Observe that no region of the ancillary graph contains entire 2-cells of the diagram. This is because the region does not contain edges of the ancillary graph in its interior; but every 2-cell of the diagram contains such edges. Thus a region of the ancillary graph contains only sectors of the 2-cells of the diagram which are cut out by the ancillary edges. The corresponding boundary sections of these 2-cells must then all identify with each other lest they violate the condition that a region of the ancillary graph cannot contain an entire 2-cell of the diagram, i.e. the boundary segments of the section of 2-cells within an ancillary region must be glued together in such a way that they form a tree. For a consistently oriented region of the ancillary graph the sectors of the 2-cells within the region will either be all positive segments or all negative segments (see figure 6).

Figure 6



If the boundary of the ancillary region is oriented counterclockwise all segments will be positive segments. If it is oriented clockwise, all segments will be negative segments. We will call the ancillary region a positive (negative) region, respectively. Figure 6 is an example of a positive region of the ancillary graph. If it can be shown that the segments within any consistently oriented region cannot be glued together then such a consistently oriented region cannot exist and we have proven that P is diagrammatically reducible.

Let $S+ = \{p_1, p_2, \ldots, p_k\}$ be the list of all positive segments that occur as cyclic subwords of the defining relation or their inverses. If there are positive segments that read the same word but occur at different places in the relations or their inverses, then they get listed in S+ once for each occurrence.

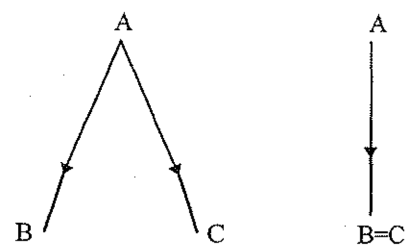
Let the list of all negative segments $S- = \{n_1, n_2, \ldots, n_l\}$ be defined analogously. Note that the inverse of a positive, negative segment is a again a positive, negative segment, respectively, and that positive and negative segments are always of even length. Let L+, L- be half the minimum length of all positive, negative segments respectively. For a positive segment p_i, let $L(p_i)$ be the initial subword of length L+ of the word p_i, and for a negative segment n_i let $L(n_i)$ be the initial subword of length L-. Let $T+ = \{L(p_i) | p_i \in S+\}$ and $T- = \{L(n_i) | n_i \in S-\}$. T+ and T- are again lists where the same word may be listed several times according to the number of occurrences.

Theorem: Let P be a presentation such that each defining relation of P contains at least one subword that is a positive segment and one subword that is a negative segment. If all $L(p_i)$ in T+ are distinct and all $L(n_i)$ in T- are distinct, then the ancillary graph of any reduced spherical diagram cannot contain consistently oriented regions which implies (by Lemma 1) that such diagrams cannot exist and hence, that P is diagrammatically reducible.

Remark: The hypothesis of the Theorem implies in particular that S+ and S- are lists of distinct words, i.e. any positive or negative segment that occurs, occurs only at one place in the relations or their inverses.

Proof: We show that their are no reduced spherical diagroms over P by contradiction. If P was such a diagram its ancillary graph would have a consistently oriented region, say positive region. This positive region consists of segments of 2 cells whose boundaries, which are positive segments are identified to a tree. This identification can be obtined by a sequence of elementary steps, where each elementary step folds a pair of edges, that already have endpoint A in common, together, see figure 7.

Figure 7



(Of course the direction of the edges could be oriented towards A.)

It is easy to see that the two edges in an elementary folding step must come from different positive segments. This come from the fact that all relations are cyclically reduced. In order for all the positive segments in a positive ancillary region to fold up entirely into a tree we claim that at least one half of some positive segment in the ancillary region must fold together with a part of an adjacent segment. For example there must be adjacent segments p_i, p_j such that the endpoint of p_i is the starting point of p_j such that either the initial half of p_j equals the initial subword of p_i^{-1} of the same length or the initial half of p_i^{-1} equals the initial subword of p_j of the same length. The proof of this claim is the same as the proof of the well known fact that a product of freely reduced words in a free group can only cancel to 1 if at least one half of one word gets cancelled by an adjacent subword. Since the diagram is reduced,the positive segments p_i^{-1} and p_j whose initial parts fold together must come from different places in the relations or their inverses, and since L+ is less than or equal to falf the length of any positive segment, it follows that there must be positive segments, that occur at different places in the relations or their inverses whose initial segments of length L+ are equal, which contradicts the hypothesis of the theorem. An analogous arguements holds for the case when the consistently oriented ancillary region is a negative region.

The following presentation is an example of the class of presentations which the theorem guarantees will be aspherical.

$$P = \langle x_1, x_2, x_3, x_4, x_5 | R_i = x_j x_i x_{i+1}^{-1} x_i x_{i+1}^{-1} x_k \rangle$$

P satisfies the condition that the T- is distinct and that S- is distinct. It further satisfies the conditions that T+ and S+ are distinct since there exists no positive segments within the presentation. This presentation can be extended by multiplying each relation by words which do not contain positive segments or negative segments within them.

Section 5: Continuing Research

A second method for determining diagrammatic reducibility shall now be developed. This method once again shall rely upon building an ancillary graph upon a diagram over the presentation, but shall not require that there exist pairs of consecutive edges which are oriented identically. We will consider presentations satisfying the following condition:

Condition Q: There exists a set Y that is a subset of the set of generators of X such that each relation contains at least two distinct elements of Y and each y_i of Y is contained in exactly two relations in the presentation.

The ancillary graph for this test shall be constructed in the following fashion. For each $y_i \in Y$ we will give each occurrence of y_i or y_i^{-1} n the relation a distinct number (called a "weight"). The number will be the same for the corresponding occurrence of y_i or y_i^{-1} in the inverse relation. Given a reduced diagram over P the ancillary graph will be constructed as follows. The vertices of the ancillary graph will be the midpoints of the 2 cells of the diagram, the ancillary edges will be the edges that are dual to the edges of the diagram labelled by elements of Y. The orientation of an ancillary edge will be determined by the weight. If the ancillary edge is dual to an edge labelled y_i and connects the midpoints of 2-cells that come together at this edges, then the weight of this edge y_i will differ in these two 2-cells (since the diagram is reduced, i.e. the 2-cells do not form a fold). The orientation of the ancillary edge will then be from the 2-cell where the edge y_i has the lesser wieght towards the 2-cell where the edge y_i has the larger weight. In order to assure that our presentation is diagrammatically reducible the ancillary graph of a reduced spherical diagram must not contain sinks, sources, or consistently oriented regions. To assure that it does not contain sinks or sources the following algorithm (Algorithm 1) may be followed in assigning weights to the elements of Y.

Algorithm 1

1. Choose a relation which contains an element of Y which has not been assigned a weight and contains no elements of Y which have been assigned a minimum weight. If all relations contain an element of Y which has been assigned a minimum weight, then choose any relation which contains an element of Y which has not been assigned a weight. Choose an element of Y within the relation that has not been assigned a weight and assign it a minimum weight. If the element exists within the relation more than once, choose the next occurrence of the element and assign it the next higher weight. Repeat this process until all occurrences of the element within the relation have been assigned a weight.

2. Next choose the relation that contains the element of Y to which you assigned a minimum in the previous step. Assign that same element a maximum in this relation. If the element exists within the relation more than once, choose the next occurrence of the element and assign it the next lower weight. Repeat this process until all occurrences of the element within the relation have been assigned a weight.

3. If this relation contains an element of Y which has not been assigned a weight proceed, otherwise go to 4. Choose another element of Y within the relation which has not been assigned a weight. Assign this element a minimum weight. Treat multiple occurrences of this element as above. Then go to 2.

4. Check to see if a relation exists that has an element of Y which has been assigned a minimum weight, but does not have an element of Y which has been assigned a maximum weight. If such a relation exists assign an element of Y within the relation a maximum weight, otherwise proceed to step 7. Treat multiple occurrences as above.

5. Next choose the relation that contains the element of Y to which you assigned a maximum in the previous step. Assign a minimum weight to this element. Treat multiple occurrences as above.

6. If this relation contains an element of Y which has not been assigned a weight proceed, otherwise go to 7. Choose another element of Y within the relation which has not been assigned a weight. Assign this weight a maximum weight. Treat multiple occurrences of this element as above. Then go to 5.

7. If all elements of Y within the presentation have been assigned a weight end, otherwise proceed to step 1.

This algorithm can be shown to prevent the existence of sinks and sources within the ancillary graph.

Proof: Assume that a source exists within the ancillary graph. This implies that all elements of Y within a relation were assigned minimum weights. Note that a relation is said to contain a minimum or maximum weight if there exists an element of Y within the relation which has been assigned a minimum or maximum weight. There are three steps within the algorithm which assign minimum weights to elements of Y. These steps are 1, 3, and 5. Step 3 assigns a minimum weight to an element of Y within a relation which received a maximum weight assignment to an element of Y in the previous step. Thus step 3 cannot be considered as contributing to a relation which only contains minimum weights. This implies that a relation which contains only minimum weights receives all weight assignments in either step 1 or step 5. First assume that a relation receives all of it's weight assignments from step 1. If one has completed step 1 followed by iterations of steps 2 and 3 and finally gets to step 4, there will exist at most one relation that satisfies the hypothesis of step 4, namely the relation to which step one is applied, but then step 4 will assign to an edge in this relation a maximum weight. This implies that a relation cannot exist that receives all of its weight assignments from step 1. Unless the relation involved in step one contains only one element of Y, which is a violation of Condition Q. Now, assume that there exists a relation which receives its weight assignments only from step 1 and step 5. We know that from steps 1-4 alone, there will exist no relation that contains only minima, but assume that a relation exists such that step 5 assigns it a minimum weight and step 6 does not assign it a maximum weight. Then step 1 could assign a minimum weight to the relation which received a minimum weight from step 5 and create a relation which only contains minimum weights. However this is a violation of the algorithm, for step 6 must apply to the relation assigned a minimum weight in step 5 if there exists an element of Y which has not been previously assigned a weight. For step 1 to assign a weight to this relation there must exist an element in the relation which step 6 assigns a weight to. Assume that there exists a relation which receives all of it's weight assignments from step 5. This implies that only one element of Y exists within the relation, for step 6 assigns a weight to the same relation that step 5 did, unless all weight assignments for the relation have been assigned. This is a violation of condition Q since all relations must contain at least two elements of Y. Therefore the algorithm prevents the existence of a source within the ancillary graph.

Assume that a sink exists within the ancillary graph. This implies that all elements of Y within a relation were assigned maximum weights. There are three steps within the algorithm which assign maximum weights. These steps are 2, 4, and 6. Steps 4 and 6 both assign maximum weights to relations which have received minimum weights in previous steps. This implies that steps 4 and 6 cannot be considered as contributing to a relation which contains only maximums. Assume that a relation exists that receives all of its weight assignments from step 2. This implies that the relation contains only one element of Y, otherwise the relation would receive a weight assignment from step 3. However, this is a violation of the presentation. Therefore the algorithm prevents the existence of a sink within the ancillary graph.

Given such an algorithm a presentation can then be shown to be diagrammatically reducible if it can be shown that the ancillary graph does not contain any consistently oriented regions. I have two hopes for conditions that shall guarantee that a consistently oriented region will not exist within the ancillary graph. The first strategy is to develop a class of presentations such that every consistently oriented path of the ancillary graph contains an element of Y interior to it, thus preventing the consistently oriented path from becoming a consistently oriented region. The second strategy is quite similar to the strategy used in the first technique. This strategy involves identifying all possible segments of relations which may lie interior to a consistently oriented region of the ancillary graph and determining characteristics of these segments which will prevent them from all being identified together.

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