Notes on Circulant Graphs and Their Isomorphisms*

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1 Definitions

Here we are concerned with a particular class of graphs known as circulant graphs (also referred to as cyclic graphs and star-polygon graphs). Our focus here is to determine when two circulant graphs are isomorphic. Throughout these notes, we will be working entirely over a finite ring $\mathbf{Z}_n$ for some number n.

Let $\mathbf{v} = (a_1, a_2, a_3, ..., a_n)$ be a vector over a field $\mathbf{F}$ and let $\mathbf{T} : \mathbf{F}^n \to \mathbf{F}^n$ be the cyclic transformation such that $\mathbf{Tv} = (a_n, a_1, a_2, ..., a_{n-1})$. Then call an $n \times n$ matrix $\mathbf{A}$ a *circulant matrix* if and only if it is of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{v} \\ \mathbf{Tv} \\ \mathbf{T}^2\mathbf{v} \\ \vdots \\ \mathbf{T}^{n-1}\mathbf{v} \end{bmatrix}, \tag{1}$$

where $\mathbf{v}$ is some row vector of n elements over $\mathbf{F}$. If the matrix $\mathbf{A}$ is of this form, it is entirely expressible in terms of the vector $\mathbf{v}$ and we will also denote $\mathbf{A}$ by $circ(\mathbf{v})$. For example, the identity matrix $\mathbf{I} = circ(1, 0, 0, 0, ..., 0)$ and the matrix for $\mathbf{T}$ is $circ(0, 1, 0, 0, 0, ..., 0)$.

Definition 1 *A* circulant graph $C_n(S)$ *is a graph with n vertices labeled 0 through $n-1$ with the following with an edge from vertex u to vertex v if and only if $v - u \in S$ where $S \subseteq \mathbf{Z}_n \backslash \{0\}$. The set S is called a* connection set *or* jump set.

Take a circulant graph $G = C_n(S)$. Then the adjacency matrix of G is the {0,1}-valued circulant matrix $circ(\mathbf{v})$ where $\mathbf{v}$ is the n-dimensional vector with

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a 1 in the i^{th} position if and only if $i \in S$. For instance, the adjacency matrix of the circulant graph $C_n\{1\}$ is the just matrix representation of $\mathbf{T}$ as defined above.

We denote the set of vertices in G by $V(G)$ and the set of edges in G by $E(G)$. In general G will be a directed graph. However if $S = -S$, then G can be regarded as an undirected graph.

2 Previous Work

Take any two circulant graphs, $G_1 = C_n(S)$ and $G_2 = C_n(T)$. Adam conjectured that $G_1 \cong G_2$ if and only if there exists an r relatively prime to n such that $rS = T$, where by rS we mean $\{rs : s \in S\}$. If $G_1 \cong G_2$ and there exists a multiplier r relatively prime to n such that $rS = T$, then G_1 and G_2 are said to be *Adam isomorphic*. If $G_1 \cong G_2$, but there is no multiplier r such that $rS = T$, we will say G_1 and G_2 are *non-Adam isomorphic*. It was noted in [2] that if an r relatively prime to n exists such that $rS = T$, then $G_1 \cong G_2$. More precisely, one isomorphism from G_1 to G_2 is the mapping which sends each $u \in V(G_1)$ to $ru \in V(G_2)$. Thus one direction of Adam's conjecture is easy to show, however the other is not true in general. Adam's conjecture was shown in [3] to hold for the case where n is square-free and fasle whenever n is divisible by 8 or an odd square. It is not yet known if Adam's conjecture holds for the remaing case where 4 but neither 8 nor an odd square divides n.

Elspas and Turner[2] provide two counterexamples to Adam's conjecture. The first example shows that the two digraphs $C_8\{1,2,5\}$ and $C_8\{1,5,6\}$ are non-Adam isomorphic. Their second example shows the two undirected graphs $C_{16}\{1,2,7,9,14,15\}$ and $C_{16}\{2,3,5,11,13,14\}$ are non-Adam isomorphic.

3 Current Work

We define a new kind of composition of circulant graphs to help us give a characterization of those graphs which are non-Adam isomorphic.

Definition 2 *Let $G_1 = C_m(S)$, $G_2 = C_n(T)$. Now let S^* denote nS and T^* denote $\{im+t : t \in T, 0 \leq i \leq n-1\}$. When $S^* \cap T^* = \emptyset$ we define the* circulant composition *of G_1 with G_2 to be*

$$C_m(S) \circ C_n(T) = C_{mn}(S^* \cup T^*).$$

Geometricially, $G_1 \circ G_2$ can be regarded as a graph of n rings[1] with m elements each where each ring has all the elements of $\mathbf{Z}_{mn}$ which fall in the same equivalence class modulo n. Then each ring is isomorphic to G_1. Furthermore,

[1] Henceforth we use the word "ring" in the geometric sense, not the traditional algebraic sense.

if we contract each ring into one vertex and disregard multiedges, then our resulting graph is isomorphic to G_2.

Now we provide several examples of non-Adam isomorphic (circulant) compositions.

$$
\begin{aligned}
C_8\{1,2,5\} &= C_4\{1\} \circ C_2\{1\} \cong \\
C_8\{1,5,6\} &= C_4\{3\} \circ C_2\{1\}. \\
C_9\{1,3,4,7\} &= C_3\{1\} \circ C_3\{1\} \cong \\
C_9\{1,4,6,7\} &= C_3\{2\} \circ C_3\{1\}. \\
C_{18}\{1,3,7,13\} &= C_6\{1\} \circ C_3\{1\} \cong \\
C_{18}\{1,7,13,15\} &= C_6\{5\} \circ C_3\{1\} \ncong \\
C_{18}\{1,6,7,13\} &= C_6\{2\} \circ C_3\{1\} \cong \\
C_{18}\{1,7,12,13\} &= C_6\{4\} \circ C_3\{1\}. \\
C_{27}\{1,3,10,19\} &= C_9\{1\} \circ C_3\{1\} \cong \\
C_{27}\{1,10,12,19\} &= C_9\{4\} \circ C_3\{1\} \ncong \\
C_{27}\{1,6,10,19\} &= C_9\{2\} \circ C_3\{1\} \cong \\
C_{27}\{1,10,15,19\} &= C_9\{5\} \circ C_3\{1\}.
\end{aligned}
$$

We will now focus on a special case of circulant compositions which encompasses our examples above. Let $G = C_m(S) \circ C_n(T)$ and suppose $n|m$. Geometrically we will regard this special case differently than the general case. Instead of looking at G as n rings of m elements, we will look at G as m rings of n elements. We will define ring i of G to be the subgraph of G comprising the vertices $i, i+m, i+2m, ..., i+(n-1)m$. Then the edges of G induced by T^* will be exactly the edges which connect each vertex in ring i to each vertex in ring j whenever $j - i \in T$.

Thus if we collapse each ring into a vertex and disregard multiedges, then our resulting graph is isomorphic to $C_m(T)$. However if we require that $T \subseteq \mathbf{Z}_n \backslash \{0\}$ then $C_m(T)$ is more limited than we would like. For this reason, we will relax our conditions on the connection set and just require that $T \subseteq \mathbf{Z}_m \backslash \{0\}$ such that $T^* \cap S^* = \emptyset$. Then we may express $C_8\{3,6,7\}$ as the composition of $C_4\{3\}$ with $C_2\{3\}$.

Now we consider isomorphisms of G. We may permute the elements in a ring i without changing any of the edges induced by T^*. Thus any isomorphism ϕ of $C_{mn}(S^*)$ which sends each vertex in ring i to a vertex in ring i in a second graph $C_{mn}(S_2^*)$ will be an isomorphism from $C_{mn}(S^* \cup T^*)$ to $C_{mn}(S_2^* \cup T^*)$. Note ϕ will map each vertex from ring i to a vertex in ring i if and only if ϕ preserves equivalences classes modulo m, i.e. for all $v \in V(G)$, $\phi(v) \equiv v \pmod{m}$.

Now we will show that $C_{mn}(S^*) \cong C_{mn}(S_2^*)$ if and only if $C_m(S) \cong C_m(S_2)$. To do this, we will show $comp(C_{mn}(S^*)) \cong comp(C_m(S))$, where by $comp(G)$ we mean the connected component of G which contains the zero element. First we show they have the same number of vertices, then we show their edge sets are isomorphic. Let us write S as $\{a_1, a_2, ..., a_j\}$ so $S^* = \{na_1, na_2, ..., na_j\}$. The number of vertices in a connected component of the circulant graph $C_m\{a_1, a_2, ..., a_j\}$ is $n/gcd(a_1, a_2, ..., a_j, n)$. Also note the graph $comp(C_{mn}\{na_1, na_2, ..., na_j\})$ has $mn/gcd(na_1, na_2, ..., na_j, nm)$, i.e. the same number of vertices. Consider the mapping ϕ of $comp(C_m(S))$ which maps each vertex v to nv. Then $\phi(comp(C_m(S)))$ will send each edge $(v, v + a_i) \in E(comp(C_m(S)))$ to the edge $(nv, nv + na_i)$ in the resulting graph. Then this resulting graph is just $comp(C_{mn}(S^*))$. Hence, $comp(C_{mn}(S^*)) \cong comp(C_m(S))$. Since the connected components of a circulant graph must be isomorphic and both graphs have the same number of connected components, we have $C_{mn}(S^*) \cong C_{mn}(S_2^*)$ if and only if $C_m(S) \cong C_m(S_2)$.

Combining our results with Adam isomorphism, we have just shown following theorem.

Theorem 1 *Suppose $C_m(S_1) \circ C_n(T)$ and $C_m(S_2) \circ C_n(T)$ are defined with n dividing m. Further suppose there exists an isomorphism from $C_m(S_1)$ to $C_m(S_2)$ which preserves equivalence classes modulo m/n. Then $C_{mn}(S_1^* \cup T^*) \cong C_{mn}(rS_2^* \cup rT^*)$ where r is any element of the multiplicative group Z_{mn}^*.*

In the case where $m = n$, we consider all integers to be equivalent modulo m/n.

Conjecture 1 *Suppose $C_n(S) \cong C_n(T)$. Then either $C_n(S)$ and $C_n(T)$ are Adam isomorphic or they are circulant compositions which are isomorphic by Theorem 1.*

References

[1] Biggs, Norman; *Algebraic Graph Theory*, Cambridge University Press (1993).

[2] Elspas, Bernard; and James Turner; Graphs with Circulant Adjacency Matrices, *J. Combinatorial Theory* **9**, 297-307 (1970).

[3] Muzychuk, Mikhail; Adam's Conjecture is True in the Square-Free Case, *J. Combinatorial Theory, Series A* **72**, 118-134 (1995).