Bundle Projections of Hyperplane Arrangements

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I. Basic Structures and Definitions

An ℓ-arrangement is a finite collection of hyperplanes in a vector space of dimension ℓ. In this paper, we will restrict our attention to vector spaces over $\mathbf{C}$. A *central arrangement* is an arrangement whose hyperplanes all contain the origin, and an arrangement which is not central is called *affine*. Unless we say otherwise, we will only be considering central arrangements. We define a *flat* $X = \cap_{H \in \mathcal{X}} H$ to be the intersection of the hyperplanes in some subset $\mathcal{X}$ of $\mathcal{A}$, and we define the *rank* of X to be its codimension in $\mathbf{C}^\ell$. This rank function gives us a matroid structure on the elements of $\mathcal{A}$, which we will call $M(\mathcal{A})$.

Let $X \vee Y = X \cap Y$, also a flat, and let $X \wedge Y$ be the unique flat of greatest rank which contains both X and Y. We consider $\mathbf{C}^\ell$ to be a flat of rank 0, therefore $X \wedge Y$ is always defined. These operators give us a lattice of flats, ordered by reverse inclusion, which we will call $L(M(\mathcal{A}))$, or simply $L(\mathcal{A})$.

Basic results of linear algebra give us the following inequality:

$$rk(X) + rk(Y) \geq rk(X \vee Y) + (X \wedge Y)$$

for any pair of flats X, Y. When equality holds, we say that X and Y form a *modular pair*. If a flat X forms a modular pair with every other flat Y, then we call X a *modular flat*.

II. $K(\pi, 1)$ Arrangements

A complex ℓ-arrangement $\mathcal{A}$ has topological structure, in addition to the combinatorial structure described above. Specifically, we are interested in the structure of $C(\mathcal{A}) = \mathbf{C}^\ell \setminus \cup_{H \in \mathcal{A}} H$, the complement of $\mathcal{A}$. It is natural to ask to what extent the combinatorial and topological structures of an arrangement determine each other. This is a question to which a complete answer is not known. It is not difficult to construct two arrangements which have nonisomorphic matroids but diffeomorphic complements [EF], but the other direction is more complicated. Randell [Ran] proves that if two arrangements with isomorphic matroids are connected by a continuous one-parameter family, then their complements are diffeomorphic. Rybnikov gives an example of two arrangements which differ by a discrete parameter, with isomorphic matroids, but whose complements he claims have nonisomorphic fundamental groups [Ryb]. This computation, however, has never been verified, nor has it been proven by any other means that such an example exists. Whether or not Rybnikov's claim proves to be correct, we would like to investigate the possibility of finding a class of examples. To do this, we introduce the property of being $K(\pi, 1)$.

A $K(\pi, 1)$ space is a space with trivial nth homotopy groups for all $n > 1$. If the complement of an arrangement is $K(\pi, 1)$, we say also that the arrangement itself is $K(\pi, 1)$. Sufficient conditions for being $K(\pi, 1)$ are given by Falk and Randell [FR], as well as by Deligne [Del], but there exist $K(\pi, 1)$ arrangements which do not fall into either of the two categories which arise from these criteria. Finding arrangements which are $K(\pi, 1)$ is a non-trivial task which is interesting for it's own sake, as well as useful in studying the relationship between the topological and combinatorial structures of an arrangement, as shown in the following

THEOREM 1 *Let $\mathcal{A}^{[3]}$ be a generic 3-dimensional slice of $\mathcal{A}$. Let $\mathcal{A}_0$, $\mathcal{A}_1$ be given with the following 3 properties, where $C_0 = C(\mathcal{A}_0)$ and $C_1 = C(\mathcal{A}_1)$:*

1) $\mathcal{A}_0$ and $\mathcal{A}_1$ are both $K(\pi, 1)$

2) $\mathcal{A}_0$ and $\mathcal{A}_1$ have lattices which are isomorphic through rank 2

3) $H^*(C_0) \not\cong H^*(C_1)$.

Then $\mathcal{A}_0^{[3]}$ and $\mathcal{A}_1^{[3]}$ have isomorphic matroids and nonisomorphic fundamental groups.

Proof. Taking a generic 3-dimensional slice preserves codimension in flats of rank less than 3, and any flat of rank 3 or greater is reduced to a point. This tells us that the lattice of $\mathcal{A}_i^{[3]}$ is completely determined by the lattice of A_i through rank 2, and since a matroid is completely determined by its lattice, $\mathcal{A}_0^{[3]}$ and $\mathcal{A}_1^{[3]}$ have isomorphic matroids. Since $\mathcal{A}_0$ and $\mathcal{A}_1$ are both Kp, their cohomologies are completely determined by their fundamental groups, thus their fundamental groups are different. By the Zariski-Lefschetz hyperplane theorem, taking a 3-dimensional slice does not have any effect on the fundamental group of a space, therefore $\pi_1(C_0) \not\cong \pi_1(C_1)$.

QED

We will now develop a method of finding $K(\pi, 1)$ arrangements.

III. Fiber Bundles

Let E, B, and F be topological spaces. A *bundle projection* is a function $f : E \to B$ such that the following two conditions hold:

1) $f^{-1}(b) \cong F \; \forall b \in B$

2) $\forall b \in B \; \exists$ an open set $U(b)$ containing b such that $f^{-1}(U(b)) \cong U(b) \times C$

If such a function exists, we call B the *base space*, F the *fiber*, and E a *fiber bundle*.

THEOREM **2** *Let $p : E \to B$ be a bundle projection, with fiber F. If F and B are both $K(\pi, 1)$, then so is E, and if B and E are both $K(\pi, 1)$, then so is F.*

Proof. Consider some $b_0 \in B$, and $f_0 \in F_0 = p^{-1}(b_0)$. Let $\imath : F_0 \hookrightarrow E$ be the inclusion map. Let $\imath_* : \pi_n(F_0) \to \pi_n(E)$ and $p_* : \pi_n(E) \to \in_n (B)$ be the functions on the nth homotopy groups induced by $\imath$ and p. Let $\Delta = \partial(p_*^{-1}) : \pi_n(B) \to \pi_{n-1}(F_0)$, where ∂ is the boundary operator. For a proof that p_*^{-1} exists, see Steenrod [Ste]. Then

$$\ldots \to^{p_*} \pi_{n+1}(B) \to^{\Delta} \pi_n(F_0) \to^{i_*} \pi_n(E) \to^{\Delta} \pi_{n-1}(F_0) \ldots \pi_2(E) \to^{p_*} \pi_2(B)$$

is an exact sequence [Ste]. It follows that if $\pi_n(F_0)$ and $\pi_n(B)$ are trivial, then so is $\pi_n(E)$, and if $\pi_{n+1}(B)$ and $\pi_n(E)$ are trivial, then so is $\pi_n(F_0)$. Since $F_0 \cong F \; \forall b_0$, this proves our theorem.

QED

Terao shows that projecting the complement of an arrangement along a flat of dimension 1 is a bundle projection [Ter], thus with the above result he verifies Falk and Randell's finding that a certain class of arrangements called *fiber type* arrangements are $K(\pi, 1)$. In this paper, we will generalize Terao's theorem to include projection along any modular flat.

First, however, in order to explicitly describe the fiber, we introduce the combinatorial notion of principal truncation.

IV. Principal Truncation

Let $\mathcal{A}$ be an arrangement, and let $M = M(\mathcal{A})$, $L = L(M)$, with rank function rk. Let X be a flat of $\mathcal{A}$. We then define the *complete principal truncation* $\bar{T}_X(M)$ with respect to X by defining a new rank function rk_T on a subset of the elements of L:

The elements of $L(\bar{T}_X(M))$ are $\{Y \mid \mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$ or $\mathcal{A}_X \subset \mathcal{A}_Y\}$, where $rk_T(Y) := rk(Y)$ if $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$, and $rk(Y) - rk(X) + 1$ if $\mathcal{A}_X \subset \mathcal{A}_Y$.

This complete principal truncation of a matroid has applications to graph theory, and it turns out also to be closely related to the matroid of the fiber arrangement by the following

THEOREM **3** *Let $\mathcal{A} = \{H_i\}$ be a complex ℓ-arrangement. Let X be a modular flat of rank r, and let $\mathcal{A}_X = \{H \in \mathcal{A} | X \subset H\}$.Let$\bar{X}$ be the subspace of $\mathbf{C}^\ell$ normal to X, and consider the function $f : \mathbf{C}^\ell \setminus \cup_{H \in \mathcal{A}} H \to \bar{X} \setminus \cup_{H \in \mathcal{A}_X} (H \cap \bar{X})$ obtained by projecting parallel to X. The poset of the affine p-arrangement $f^{-1}(\vec{v})$, for any $\vec{v}$, is obtained by deleting all flats containing X from $L(\bar{T}_X(M))$. Note that this arrangement is not central.*

Proof. Let $\vec{v}$ be an element of $\bar{X} \setminus \cup_{H \in \mathcal{A}_X} (H \cap \bar{X})$. Then the arrangement $p^{-1}(\vec{v})$ lies in the space $S(\vec{v}) = \{\vec{v}\} + X$, and its hyperplanes are $\{K_{i_j}(\vec{v}) = H_{i_j} \cap S(\vec{v}) \mid H_{i_j} \notin \mathcal{A}_X\}$. Let

$Y = \cap_{H \in \mathcal{A}_Y} H = \cap_{j=1}^{a} H_{i_j}$ be a flat of $\mathcal{A}$. We will show that $Y'(\vec{v}) = \cap_{j=1}^{a} K_{i_j}(\vec{v}) = \emptyset$ if $\mathcal{A}_Y \cap \mathcal{A}_X \neq \emptyset$, and that if $\mathcal{A}_Y \cap \mathcal{A}_X = \emptyset$, then $codim_{\mathbf{C}^p}(Y'(\vec{v})) = codim_{\mathbf{C}^\ell}(Y)$. This will tell us that the poset $f^{-1}(\vec{v})$ consists of the flats of $\mathcal{A}$ which are disjoint from X, with their ranks unchanged. These are exactly the flats of $L(\bar{T}_X(M))$ not containing X.

If $\exists H_k \in \mathcal{A}_Y \cap \mathcal{A}_X$, then $Y'(\vec{v}) = Y \cap S(\vec{v}) = H_k \cap Y \cap S(\vec{v}) = \emptyset$, because $H_k \cap S(\vec{v}) = \emptyset$. We are now left with only the case where $\mathcal{A}_Y \cap \mathcal{A}_X = \emptyset$, which we will assume from this point on.

Lemma 1. *Y has rank at most p.*

Proof. By hypothesis, $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$. Since X and Y form a modular pair, $rk(X) + rk(Y) = rk(X \vee Y) + rk(X \wedge Y)$. Then $r + rk(Y) = rk(X \wedge Y) \leq \ell$, therefore $rk(Y) \leq \ell - r = p$.

QED

Lemma 2. *Let U and W be subspaces of a vector space $V = U + W$. Let $v_0 = u_0 + w_0$ be given, $u_0 \in U$ and $w_0 \in W$. Then $(\{v_0\} + U) \cap W = \{w_0\} + (U \cap W)$.*

Proof. $\{v_0\} + U = \{w_0\} + U$, and the result follows trivially.

QED

Let $q = rk(Y)$. Since $\mathcal{A}_X$ and $\mathcal{A}_Y$ are disjoint, $X + Y = \mathbf{C}^\ell$. Then Lemma 2 states that $Y'(\vec{v}) = Y \cap S(\vec{v})$ has the dimension of $X \cap Y$. Then $codim_{\mathbf{C}^p}(Y'(\vec{v})) = p - dim(X \cap Y) = p - dim(X \wedge Y) = p + dim(X \vee Y) - dim(X) - dim(Y) = p + \ell - p + q - \ell = q = codim_{\mathbf{C}^\ell}(Y)$.

QED

Corollary. *For all* $\vec{v}, \vec{w} \in \bar{X} \setminus \cup_{H \in \mathcal{A}_X} (H \cap \bar{X})$, $f^{-1}(\vec{v}) \cong f^{-1}(\vec{w})$.

Proof. Since $\bar{X} \setminus \cup_{H \in \mathcal{A}_X} (H \cap \bar{X})$ is path connected, $f^{-1}(\vec{v})$ and $f^{-1}(\vec{w})$ are joined by a one-parameter family. Explicitly, let $\gamma : [0,1] \to in\bar{X} \setminus \cup_{H \in \mathcal{A}_X} (H \cap \bar{X})$ be a path connecting $\vec{v}$ and $\vec{w}$. Then $f^{-1}(\gamma(t))$ gives us the necessary family. Then by Randell's theorem [Ran], $f^{-1}(\vec{v}) \cong f^{-1}(\vec{w})$.

QED

Thus we have proven the first of the two necessary conditions for f to be a bundle projection. It should be intuitively clear that the second condition is also true, since all of the spaces with which we are dealing are linear. The proof of this condition, however, is still in progress. In the following section, in which we explore the possible applications of Theorem 2, we will assume that we have completed the proof that f is a bundle projection.

V. Examples

Let M_1 be a matroid on some set E_1, and M_2 a matroid on E_2. Let $E = E_1 \cap E_2$ and assume that we have a matroid $N = M_1|E = M_2|E$, such that N is modular in M_1. Then we define the *generalized parallel connection* $P_N(M_1, M_2)$ of M_1 and M_2 along N to be the matroid on the set $E_1 \cup E_2$ such that $P_N(M_1, M_2)$ has a flat X on $E_X \subset E_1 \cup E_2$ if and only if M_1 has a flat on $E_X \cap E_1$ and M_2 has a flat on $E_X \cap E_2$, where $rk(X) := rk(X \cap M_1) + rk(X \cap E_2) - rk(X \cap N)$. Then we have the following

Corollary. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrangement realizations of M_1 and M_2, respectively, and let $\mathcal{A}$ be a realization of $M = P_N(M_1, M_2)$. Then $C(\mathcal{A})$ is a fiber bundle over $C(\mathcal{A}_2)$.*

Proof. As given in Oxley [Ox], M_2 is modular in M, and then Theorem 3 can be applied.

QED

Again, we have proven this corollary to the extent that we have proven Theorem 3: we have proven that the preimage of the projection from $C(\mathcal{A})$ to $C(\mathcal{A}_2)$ is constant over the elements of $C(\mathcal{A}_2)$ up to diffeomorphism, which is only a small step from the complete result. We now give a second method of proof of this corollary, which provides insight into the structure of the generalized parallel connection, but depends on Theorem 3 having been proven in its entirety:

Proof. Let f_1 and f_2 be the respective projections from $C(\mathcal{A}_1)$ and $C(\mathcal{A}_2)$ to $\bar{X}$, where f_1 is a fiber projection. Then C(A) is the fibered product [Ste] of f_1 and f_2, i.e. $C(\mathcal{A})$ is isomorphic to the space $\{(\vec{v}_1, \vec{v}_2) \in C(\mathcal{A}_1) \times C(\mathcal{A}_2) \mid f_1(\vec{v}_1) = f_2(\vec{v}_2)\}$. It then follows immediately that $f_1' : C(\mathcal{A}) \to C(\mathcal{A}_2)$ taking $(\vec{v}_1, \vec{v}_2)$ to $\vec{v}_2$ is a bundle projection.

QED

This result provides a method of constructing $K(\pi, 1)$ arrangements. More generally, with Theorem 3 and its corollary, we have developed a test for $K(\pi, 1)$-ness. The following is an example of how this test can be applied.

Let $\mathcal{A}$ be the $\mathbf{C}^4$ arrangement consisting of the hyperplanes $x = 0$, $y = 0$, $x = \pm y$, $x = \pm z$, $y = \pm z$, $x = \pm w$, $y = \pm w$, and $z = \pm w$. The intersection X of the first 4 hyperplanes listed is a modular flat. Projecting along X and taking the inverse image of a point (x_0, y_0), we get the 2-dimensional affine arrangement consisting of the hyperplanes $z = \pm x_0$, $z = \pm y_0$, $w = \pm x_0$, $w = \pm y_0$, where $0 \neq x_0 \neq y_0 \neq 0$. This fiber is shown to be $K(\pi, 1)$ by Falk and Randell [FR], and the base is simplicial [Del], therefore $K(\pi, 1)$. Thus $\mathcal{A}$ is $K(\pi, 1)$.

As a second example, consider the arrangement $\mathcal{A}'$ obtained by deleting the hyperplanes $x = -z$, $x = -w$, $y = -z$, $y = -w$ from $\mathcal{A}$. In our new arrangement, the same flat is modular, but projection along X now gives the fiber $z = x_0$, $z = y_0$, $w = x_0$, $w = y_0$, which is not $K(\pi, 1)$ because it contains a generic line [Falk]. The base space of this projection is the same as in our

previous example, and therefore is $K(\pi, 1)$. Thus THEOREM 1 tells us that $\mathcal{A}'$ is not $K(\pi, 1)$.

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