Degrees of Differentiability*

Peter E. Devlin

Department of Mathematics

Manhattan College

Riverdale, New York

pdevlin@student.manhattan.edu

advisor: John Hagood, Ph.D

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INTRODUCTION

The concept of a degree of differentiability arises from a study of functions which are not differentiable (either everywhere, at one point or at several points), but are differentiable when raised to some power, say α .

Definition 1. *A function f is said to have a degree of differentiability of β at a point x if f^α is differentiable at x for every $\alpha > \beta$, but not for $\alpha < \beta$. f^β may or may not be differentiable.*

A study of functions which possess a degree of differentiability yields many interesting results, and demonstrates some of the special properties and characteristics that such functions possess. Such a study also provides numerous examples, which serve to shed light on the topic, as well as provide new avenues for further investigation. The following study attempts to develop some basic principles surrounding functions having such a degree of differentiability, it presents several examples, and opens up many other questions pertaining to the topic.

BACKGROUND

The study of differentiation has yielded numerous examples of functions which fail to be differentiable. The level of nondifferentiability ranges from, in very simple cases, a single point to much more complex cases including nondifferentiability everywhere. To set the stage for a study of such functions it is helpful to recall what exactly is meant by differentiation.

Definition 2. *Let f be defined on the open interval (a, b), and assume that $x_0 \in (a, b)$. Then f is said to have a derivative at x_0 whenever the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. This limit, denoted by $f'(x_0)$ is called the derivative of f at x_0.

The limit in Definition 2 exists if and only if both the right hand limit ($\lim_{x \to x_0^+}$) , and the left hand limit ($\lim_{x \to x_0^-}$) , exist and these limits are equal. From this definition it is not hard to come up with functions which fail to be differentiable somewhere. It is helpful to recall here that a function which is differentiable at a point is necessarily continuous at that point, yet it is possible for continuous functions to fail to have a derivative. This is stated as Theorem 1.

Theorem 1. *If f has a derivative at a point x_0 in (a,b) , then f is continuous at x_0.*

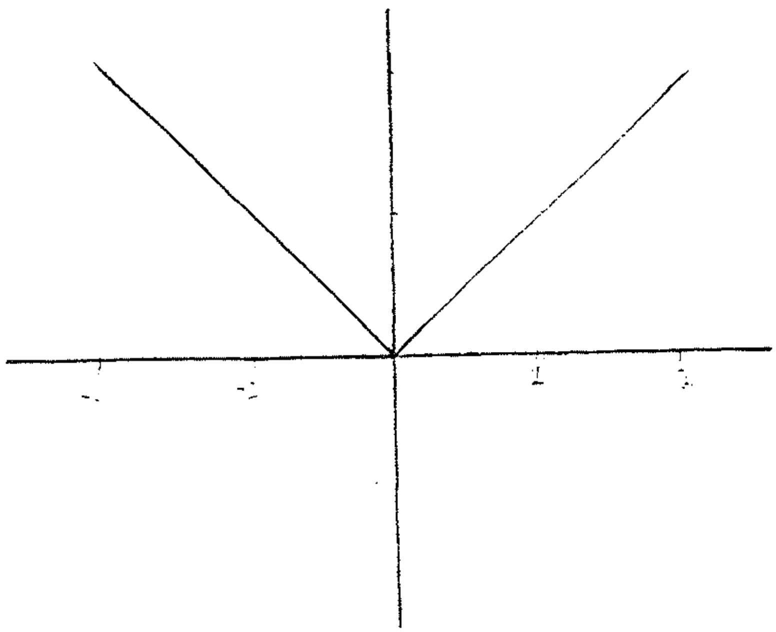
The converse of Theorem 1 need not be true.

At this point it is helpful to look at some examples of functions which fail to have a derivative.

Example 1.

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{otherwise.} \end{cases}$$

The graph of this function has a sharp "corner" at the point $x_0 = 0$. Thus it should be obvious that the derivative will not exist at 0. Note, that f(x) is continuous everywhere.



Consider the difference quotient

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Here,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x}{x} = 1$$

Thus, the one-sided limits are different and we conclude that $f'(0)$ does not exist.

Example 1 gives a function which is not differentiable at a single point, 0. Functions which are not differentiable at a point, but are continuous at that point may arise from a "corner,".as in Example 1, or a "cusp," as with $f(x) = x^{\frac{2}{3}}$. Other functions, such as those introduced by Weierstrass and Hildebrandt, are continuous everywhere, yet have a derivative nowhere. One such function can be thought of as a "saw-tooth" function.

Example 2. **"Saw-tooth"** Let $\{x\}$ denote the distance from x to the nearest integer. Define f_n by:

$$f_n(x) = \frac{1}{10^n} * \{10^n x\}.$$

Then $|f_n(x)| \leq \frac{1}{10^n}$ $\forall x$ and since $\sum_{n=1}^{\infty} \frac{1}{10^n}$ converges, $\sum_{n=1}^{\infty} f_n$ converges uniformly. Thus, since each f_n is continuous, this implies that

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{10^n} * \{10^n x\}$$

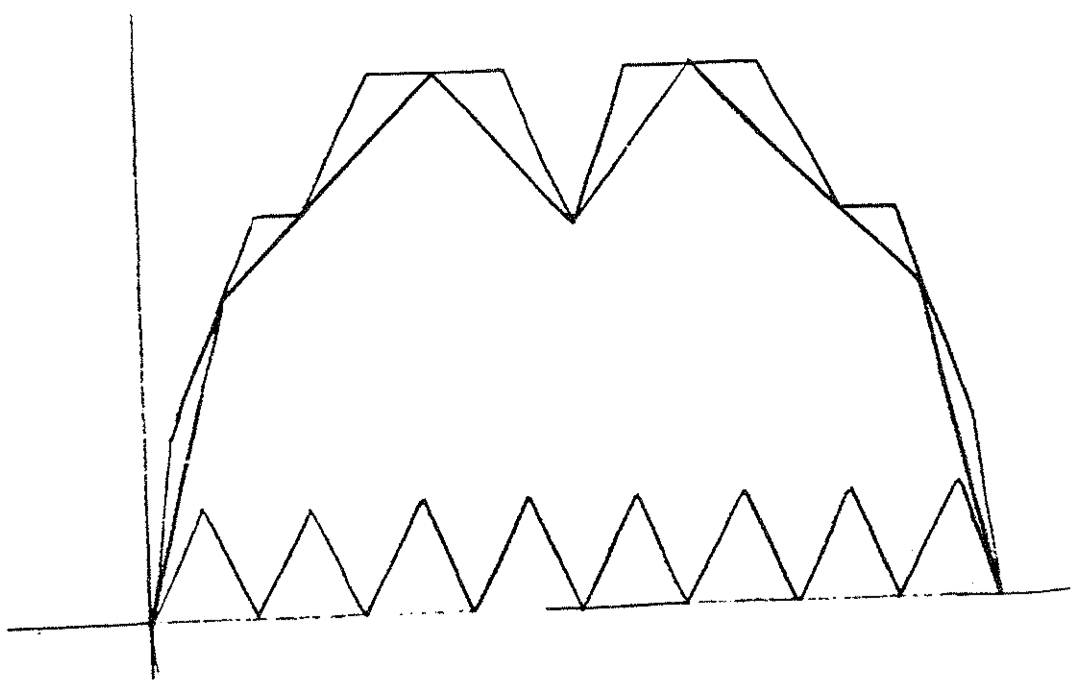
is also continuous. In fact,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{10^n} * \{10^n x\}$$

is continuous everywhere and differentiable nowhere. The proof that f is differentiable nowhere is given in [1].

The functions mentioned in Examples 1 and 2 are drawn from an extremely large class of functions. In fact, "most" continuous functions have the property of being differentiable nowhere, and indeed these functions form a set of second category [2].

Partial Sums of f_n



PROPERTIES OF FUNCTIONS WITH DEGREE β

We now turn our attention to functions which possess a degree of differentiability as specified in Definition 1. To begin, we examine a simple example.

Example 3. Again we consider

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{otherwise.} \end{cases}$$

We have seen that this function is not differentiable at the point $x_0 = 0$. However, what happens when f is raised to some power, say α. As a conjecture, we propose that for large enough powers, a smoothing of the graph will occur. Let

$$f^\alpha(x) = |x|^\alpha = \begin{cases} x^\alpha, & \text{if } x \geq 0; \\ (-x)^\alpha, & \text{otherwise.} \end{cases}$$

Again we examine the difference quotient

$$\lim_{x \to x_0} \frac{f^\alpha(x) - f^\alpha(x_0)}{x - x_0}$$

Here,

$$\lim_{x \to 0^-} \frac{f^\alpha(x) - f^\alpha(0)}{x - 0} = \lim_{x \to 0^-} \frac{(-x)^\alpha}{x} = (-x)^{\alpha-1}$$

$$\lim_{x \to 0^+} \frac{f^\alpha(x) - f^\alpha(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^\alpha}{x} = x^{\alpha-1}$$

It is now clear that for $\alpha > 1$ both one-sided limits exist and are equal to 0. Here, we can conclude that our function has a degree of differentiability of $\beta = 1$. It is now permissible to write, for $\alpha > 1$,

$$(f^\alpha(x))' = \begin{cases} \alpha * x^{\alpha-1}, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -\alpha * (-x)^{\alpha-1}, & \text{otherwise.} \end{cases}$$

A similar analysis of the function $g(x) = x^{\frac{2}{3}}$ (which has a "cusp" at $x_0 = 0$) reveals that $g(x)$ has a degree of differentiability of $\beta = \frac{3}{2}$.

Now that the concept of a degree of differentiability has been established, it seems natural to consider what possible values this degree can obtain. It is important to note that only positive powers are being considered.

Theorem 2. *There are continuous functions of any positive degree of differentiability.*

Proof

Take $f(x) = |x|^{\frac{1}{p}} \qquad p > 0$

In order for $f^\alpha(x)$ to be differentiable at $x_0 = 0$ it is necessary and sufficient that

$$\lim_{x \to 0} \frac{(|x|^{\frac{1}{p}})^\alpha}{x}$$

exists.

Thus. $\lim_{x\to 0} |x|^{\frac{\alpha}{p}-1}$ must exist. This happens when

$$\frac{\alpha}{p} - 1 > 0.$$

Thus, the necessary and sufficient condition is that $\alpha > p$.

Now, in order to create a function of degree $\tau > 0$. let $p = \tau = p$ above.

Any $\alpha > \tau$ will cause $f^\alpha(x)$ to be differentiable, while any $\alpha < \tau$ will cause f^α to be non-differentiable at 0.

We have thus shown that any degree of differentiability is possible. Now, suppose the degree of differentiability is already known for a given function $f(x)$. What can be said about other powers of $f(x)$?

Theorem 3. *Let $f \geq 0$ and $\alpha > 0$. If $(f^\alpha)'$ exists at x, then $(f^\lambda)'$ exists at x for every $\lambda > \alpha$.*

Proof

Assume that f^α is differentiable at x. Assume $\lambda > \alpha$.

Then $f^\lambda = f^{\alpha * \frac{\lambda}{\alpha}} = (f^\alpha)^{\frac{\lambda}{\alpha}}$. This is differentiable at x by the chain rule.

[In fact. $(f^\lambda(x))' = [(f^\alpha(x))^{\frac{\lambda}{\alpha}}]' = \frac{\lambda}{\alpha}(f^\alpha(x))^{\frac{\lambda}{\alpha}-1} * (f^\alpha(x))'$

Thus, we are left with $(f^\lambda(x))' = \frac{\lambda}{\alpha}(f^\alpha(x))^{\frac{\lambda-\alpha}{\alpha}} * (f^\alpha(x))' = \frac{\lambda}{\alpha}(f^{\lambda-\alpha}(x)) * (f^\alpha(x))'$

It is known that $\lambda > \alpha$, so we also know that $\lambda - \alpha > 0$

thus $(f^{\lambda-\alpha}(x))$ exists $\forall x$, while $(f^\alpha(x))'$ exists by hypothesis.]

Therefore. $(f^\lambda(x))'$ exists $\forall \lambda > \alpha$.

Theorems 2 and 3 serve to shed some light on the degree of the functions being considered. Now we will shift our attention to the points of nondifferentiability. Obviously, for a function which has a positive degree of differentiability, "bad spots" are "smoothed" by an appropriate α. There are also certain conclusions which can be drawn about the points x_0 of nondifferentiability, based on the knowledge that f has some degree of differentiability. For a function with a certain degree of differentiability, points at which f is not differentiable possess certain inherent properties. The first of which is stated in Theorem 4.

Theorem 4. *Assume $f \geq 0$, and $\alpha > 0$. If f^α is differentiable at x and $f(x) > 0$, then. f is differentiable at x.*

The theorem states that if a function f, has a positive value at some point x, and some power of the function is differentiable at x, then the function itself must be differentiable at the point x. The theorem results from the fact that "bad" spots, which do not lie on the axis can never be smoothed by a simple power operation. In fact such spots are worsened when raised to a power.

Proof

Since $f(x) = (f^\alpha(x))^{\frac{1}{\alpha}}$ and $f^\alpha(x) > 0$, f is differentiable at x by the chain rule.

Corollary. *There exist functions which do not have a finite degree of differentiability.*

Such a function can easily be produced, based on Theorem 4. Define$f \geq 0$ such that f is not differentiable at a point x_0, with $f(x_0) > 0$. Clearly $f^\alpha(x_0)$ will not be differentiable for any α.

Theorem 4 provides us with the knowledge of where the points of nondifferentiability must lie, but it leaves open the question of just how many "bad points" are possible. That is, given some degree of differentiability, say β, can there be a continuous function f on $[a, b]$ for which f maintains the degree of differentiability on any nondegenerate subinterval of $[a, b]$? In order for this to occur, the points of nondifferentiability would need to be dense in $[a, b]$.

What we are looking for is a function $f : [a, b] \to [0, \infty)$ with f continuous, and for some $\beta > 0$, f^α differentiable for $\alpha > \beta$, but on each interval $I \subseteq [a, b]$ and for each $\alpha < \beta$, f^α is not differentiable somewhere on I.

No such function exists, as is stated in Theorem 5. In fact, any such function would necessarily be identically zero.

Theorem 5. *Let $\beta \geq 0$ and let f have degree of differentiability β on each interval $I \subseteq [a, b]$. Then either $\beta = 0$ or $f \equiv 0$ on $[a, b]$.*

__Proof__

Assume that $\beta > 0$. Each interval $I \subseteq [a, b]$ must contain a point x such that f^α is differentiable at x whenever $\alpha > \beta$, but not for $\alpha < \beta$. This forces $f(x) = 0$ at such points. Thus, every interval $I \subseteq [a, b]$ contains a zero of f. Pick $y \in [a, b]$ and generate a sequence $\{y_n\}$ with $f(y_n) = 0$ for each n by

selecting $y_n \in (y - \frac{1}{n}, y + \frac{1}{n}) \cap [a, b] \ni f(y_n) = 0$. Then, $|y_n - y| < \frac{1}{n}$, so $y_n \to y$.

By continuity $f(y) = \lim_{n \to \infty} f(y_n) = 0$

The points of nondifferentiability of a function with positive degree of differentiability therefore cannot be dense on any interval, without forcing the function to be identically zero, and consequently differentiable everywhere on that interval.

We have seen in the Corollary to Theorem 4 that there exist functions of an infinite degree of differentiability. These are functions such that f^α is not differentiable for any $\alpha > 0$. It may seem counterintuitive to consider the existence of a function which has such an infinite degree of differentiability on some closed interval $[a, b]$, yet has a finite degree of differentiability on any closed subinterval of the interior (a, b). This function would by definition never become differentiable when raise to any power $\alpha > 0$, but considering the function on a subinterval would produce differentiability for some such α.

FUNCTIONS

Function 1

Given an interval $[a, b]$. There exist continuous functions on $[a, b]$ such that on each subinterval $[c, d] \subset [a, b]$, the function is of finite degree of differentiability, but the function is not of finite degree on $[a, b]$.

Define

$$x_0 = 0.$$

$$x_n = x_{n-1} + \sum_{j=n-1}^{n} \frac{1}{2^{(j)^2}} \quad for \quad n \geq 1,$$

$$f_n(x) = \left[\left(\frac{1}{2^{(n)^2}} \right)^2 - (x - x_n)^2 \right]^{\frac{1}{n}}$$

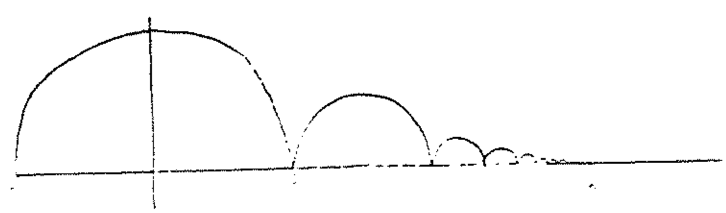
on $\left[x_n - \frac{1}{2^{n^2}}, x_n - \frac{1}{2^{n^2}} \right]$ and 0 otherwise.

$$f = \sum_{n=0}^{\infty} f_n(x) \quad on \ [-1, k].$$

and

$$k = \sum_{n=0}^{\infty} \frac{1}{2^{n^2}}.$$

Each f_n has a hump-like graph. The heights of the humps approach zero. The function f has a finite degree of differentiability on any closed subinterval of $(-1, k)$, but f does not have finite degree of differentiability on $[-1, k]$. The reason for this is that there are an infinite number of humps, connecting at points of nondifferentiability. Each consecutive point of nondifferentiability requires a larger α than the one before. In particular , $\alpha = n$ for f_n. Thus on any subinterval of $[-1, k]$ we can take α to be the α required for the last point of nondifferentiability in that subinterval. Since, $[-1, k]$ has no such last point, no α is acceptible.



Function 2: the modified ruler

The idea of a degree of non-differentiability is employed in Darst and Taylor's article "Differentiating Powers of an Old Friend" [3]. The article deals with the well known "ruler" function, and its powers. The ruler function is defined by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q}; \\ 0, & \text{otherwise.} \end{cases}$$

with p and q relatively prime integers and $q > 0$.

The ruler function is continuous at zero and at all irrationals. It is differentiable nowhere.

Powers f^α of f are defined for $\alpha > 0$ by $f^\alpha = [f(x)]^\alpha$.

Theorem (Darst and Taylor). *If $1 < \alpha \leq 2$, then f^α is differentiable only at zero. If $\alpha > 2$, then f^α is differentiable almost everywhere.*

The proof can be found in [3].

Through a relatively simple alteration, a modified ruler function can be produced such that the function, call it g, behaves similarly to the original ruler function, yet no power of g is differentiable.

Let us define $g : [0, 1] \to \Re$ as follows:

$$g(x) = \begin{cases} \frac{1}{q}^{\frac{1}{\sqrt{\log q}}}, & \text{if } x = \frac{p}{q}; \\ 0, & \text{otherwise.} \end{cases}$$

with p and q relatively prime integers and $q > 0$, and $\log q = \log_2 q$.

Theorem. *$g(x)$ is nowhere differentiable, and g^γ is not differentiable for any $\gamma > 0$.*

Proof

First we must show that g is differentiable nowhere.

Without loss of generality, consider $0 \leq x \leq 1$. Since g is not continuous on the set of nonzero rationals, the only possible points of differentiability are the irrationals and zero.

case 1: $x = 0$

Look at,

$$\frac{[g(x+h) - g(x)]}{h} = \frac{g(h)}{h}$$

and let $\{h_i\}$ be a sequence of nonzero irrationals having zero as limit.

Then,

$$\lim_{i \to \infty} \frac{g(h_i)}{h_i} = 0$$

Now, let $h_i = (\frac{1}{i})$ $\quad i = 1, 2, 3, \ldots.$

This yields,

$$\frac{g(h_i)}{h_i} = \frac{(\frac{1}{i})^{\frac{1}{\sqrt{\log i}}}}{(\frac{1}{i})} \to \infty$$

Therefore, $\lim_{h \to 0} \frac{g(h)}{h}$ does not exist, and g is not differentiable at $x = 0$.

case 2: x is an irrational number

We have,

$$\frac{[g(x+h) - g(x)]}{h} = \frac{g(x+h)}{h}$$

Let $\{h_i\}$ be a sequence of nonzero real numbers having zero as limit, and such that $x + h_i$ is irrational for each i.

Then,

$$\lim_{i \to \infty} \frac{[g(x+h_i)]}{h_i} = 0$$

Now, express x as $.a_1 a_2 a_3 \ldots . a_n \ldots .$

Choose $h_i = 0.a_1 a_2 \ldots . a_i - x$. Since $x \neq 0$ and $a_i \neq 0$ for some i,

we may pick N to be the least integer such that $N > 1$ and $a_N \neq 0$. Let p and q be relatively prime integers such that $0.a_1 a_2 \ldots . a_i = p/q$ Then $2^N \leq q \leq 10^i$ whenever $i \geq N$.

Thus,

$$g(x+h_i) = g(0.a_1a_2....a_i) \geq 10^{\frac{-i}{\sqrt{N}}} \; \forall i \geq N, \quad and\, |h_i| \leq 10^{-i}.$$

In this case,

$$\frac{[g(x+h_i)]}{h_i} \to \infty$$

Thus,

$$\lim_{h \to 0} \frac{[g(x+h)]}{h}$$

does not exist, and g is not differentiable at each irrational number.

Therefore, by cases g is differentiable nowhere.

Now it remains to show that no power of g is differentiable.

At $x = 0$,

$$\frac{g^{\alpha}(1/i)}{1/i} = i^{1-\alpha/\sqrt{\log i}} \to \infty$$

as $i \to \infty$.

For any irrational number $x \in (0, 1]$ and any $\alpha > 0$, pick $N > \alpha^2$ such that $a_N \neq 0$ in the decimal expansion of x. Then as above,

$$\frac{g^{\alpha}(x+h_i) - g^{\alpha}(x)}{h_i} = \frac{g^{\alpha}(x+h_i)}{h_i} \geq 10^{i(1-\alpha/\sqrt{N})} \to \infty$$

as $i \to \infty$. Therefore, g^{α} is not differentiable for $\alpha > 0$.

Note that unlike our previous examples, the ruler function and modified ruler function are not continuous everywhere. They are however continuous almost everywhere, which means that they are continuous except on a set of Lebesgue measure 0.

Function 3: A Cantor Function

The Cantor set, proposed by Georg Cantor (1845-1918), has become one of the richest sources of examples in analysis. The set possesses the properties of being uncountable, nowhere dense, and perfect. We will present a function defined on the Cantor set, but first we shall give some background.

The Cantor Set

Let

$$C_1 = [0,1]\backslash(\frac{1}{3},\frac{2}{3})$$

$$C_2 = C_1\backslash[(\frac{1}{9},\frac{2}{9}) \cup (\frac{7}{9},\frac{8}{9})]$$

$$\bullet$$

$$\bullet$$

$$\bullet$$

$$C_n = C_{n-1}\backslash[(\frac{1}{3^n},\frac{2}{3^n}) \cup \bullet\bullet\bullet \cup (\frac{3^n-2}{3^n},\frac{3^n-1}{3^n})]$$

Each stage of the creation of the set results from removing the open middle thirds of each interval remaining in the previous stage.

$$The\ Cantor\ Set\ C = \cap_{n=1}^{\infty} C_n.$$

The Cantor set is uncountably infinite. It is a measurable set, with $\mu(C) = 0$, where μ represents Lebesgue measure. The length of the removed intervals is $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$.

We can make the following observations about the Cantor set: (a) It is a closed set. (b)The Cantor set is nowhere dense in $[0,1]$. This means that C contains no interval of $[0,1]$. (c)Also, the Cantor set is Perfect. This means that every point of C is a limit point of C.

For proofs of the above statements, see [4].

In working with the Cantor set it is often helpful to consider the ternary representation of elements of C. In considering this representation, it is seen that every point $x \in C$ can be represented in exactly one way by a series of the form

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

where each a_n is either 0 or 2. Every number represented in this way is in C. This ternary representation can be used to prove that the Cantor set is uncountable.

Ternary notation can be used to figure out where a point x in the Cantor set is located. To do this we begin at zero and move to the right as follows. Now, any zero in the ternary expansion of x will represent a time when we stay where we are, and any two in the n^{th} place will represent a jump to the next consecutive right endpoint of intervals in C_n. Thus..2. corresponds to the point $\frac{2}{3}$. Any terminating expansion will be an endpoint. Nonterminating expansions, except those ending in repeating 2's, will not be endpoints.

Now that we have some understanding of the Cantor set, let us consider an extension to degrees of differentiability. Suppose we were to place a zero point on the x-axis at every point of the Cantor set. This would produce a function with an uncountable number of "bad spots", but the fact that the Cantor set is nowhere dense tells us that the function does not have to be identically zero.

The following example employs absolute continuity, and bounded variation. For a complete summary of these concepts see [5].

Let f be a real valued function defined on the interval $[a, b]$, and let $a = x_0 < x_1 < \ldots\ldots < x_k = b$ be any subdivision of $[a, b]$. Define

$$P = \sup \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$

$$N = \sup \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

$$T = \sup \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

over all possible subdivisions.

Here, ρ^+ denotes ρ if $\rho \geq 0$ and 0 if $\rho \leq 0$, while $\rho^- = |\rho| - \rho^+$.

Definition (Royden): We say that f is of bounded variation over $[a, b]$ if $T_a^b < \infty$

Definition (Royden):A function f defined on $[a, b]$ is said to be Absolutely Continuous on $[a, b]$ if, for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$$

for every finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

A Cantor Function: An example of $f : [0,1] \to \Re$ such that f has degree of differentiability β and $E' = f^{-1}(\{0\})$ is uncountable, with $\mu(E') = 0$.

Let $E' = C$ denote the Cantor set. Let each $x \in E'$ be a point where $f(x) = 0$. We want the endpoints of each removed interval be the points of nondifferentiability.

Now let $c < d$ denote the endpoints of some removed interval, and define

$$f|_{[c,d]}(x) = [(x-c)(d-x)]^{\frac{1}{\beta}}$$

For each removed interval $(a_{k,n}, b_{k,n})$, $f \in BV\ [a_{k,n}, b_{k,n}]$ and f has a continuous derivative on $(a_{k,n}, b_{k,n})$, so f' is Riemann integrable on closed subintervals of $(a_{k,n}, b_{k,n})$ and Lebesgue integrable (also, improper Riemann integrable) on $[a_{k,n}, b_{k,n}]$. By taking limits, it is easy to see that $f \in AC\ [a_{k,n}, b_{k,n}]$ and the Fundamental Theorem of Calculus holds. Considering the whole interval, f is bounded, since each hump is bounded by 1, and f is measurable since f is the limit of the sequence of measurable humps.

Consider the removed interval (c, d) .

$$f'|_{[c,d]} = 0 \ when \ x = \frac{c+d}{2}$$

This shows us that, as was to be expected, the maximum height of each hump is attained at its midpoint.

At this point, the height of the hump is given as:

$$f(\frac{c+d}{2}) = 4^{\frac{-1}{\beta}}[(d-c)^2]^{\frac{1}{\beta}}$$

Now, because f is absolutely continuous on $[c, d]$,

$$\int_c^d |f'| = T_c^d(f) = P_c^d + N_c^d = [\frac{(d-c)^2}{4}]^{\frac{1}{\beta}} + [\frac{(d-c)^2}{4}]^{\frac{1}{\beta}} = 2[\frac{(d-c)^2}{4}]^{\frac{1}{\beta}}$$

Let $< E_i >$ consist of the subintervals of $[0,1]$ removed to from the Cantor set and put $E = \cup E_i$. Note that $\int_0^1 |f'| = \int_E |f'|$ since $\mu(E') = 0$. Since $|f'|$ is a nonnegative measurable function and $< E_i >$ consists of disjoint measurable sets, and since $f \in AC$ on each $[a_{k,n}, b_{k,n}]$,

$$\int_0^1 |f'| = \sum \int_{E_i} |f'| = \sum T_{a_{k,n}}^{b_{k,n}}(f)$$

Also,

$$\sum T_{a_{k,n}}^{b_{k,n}}(f) = \sum_{n=1}^{\infty} 2^n \left[\frac{9^{-n}}{4}\right]^{\frac{1}{\beta}} = 4^{-\frac{1}{\beta}} \sum_{n=1}^{\infty} \left(\frac{2}{9^{\frac{1}{\beta}}}\right)^n$$

This series converges for $9^{\frac{1}{\beta}} > 2$.

This means that f has bounded variation for $\beta < \log_2 9$.

We want a function which works for all $\beta > 0$. A simple scaling of f results in such a function.

So instead, for each removed interval (c, d), let

$$f(x)|_{[c,d]} = \frac{1}{2^n}[(x-c)(d-x)]^{\frac{1}{\beta}}.$$

At this point, the height of each hump at the n^{th} stage is given as:

$$f\left(\frac{c+d}{2}\right) = \frac{1}{2^n}\left[\frac{(d-c)^2}{4}\right]^{\frac{1}{\beta}}.$$

Again, we have

$$\int_0^1 |f'| = \sum \int_{E_i} |f'| = \sum T_{a_{k,n}}^{b_{k,n}}(f) = 4^{-\frac{1}{\beta}} \sum_{n=1}^{\infty} \left[\frac{1}{9^n}\right]^{\frac{1}{\beta}} < \infty$$

for any $\beta > 0$.

This tells us that f is of bounded variation over [a,b], and thus f' is integrable, with

$$\int_0^x f' = \sum \int_{[0,x]\cap E_i} f'.$$

Also, each x can be an element of at most one E_i, with

$$\int_{[0,x]\cap E_i} f' = 0 \quad if \quad x \notin E_i.$$

This is due to the fact that the Fundamental Theorem of Calculus applies on each E_i.

Suppose $x \in E_{i_0}$. Then

$$\int_0^x f' = \sum \int_{[0,x]\cap E_i} f' = \int_{[0,x]\cap E_{i_0}} f' = \int_{a_{k,n}}^x f'$$

where $a_{k,n}$ is the left endpoint of E_{i_0}.

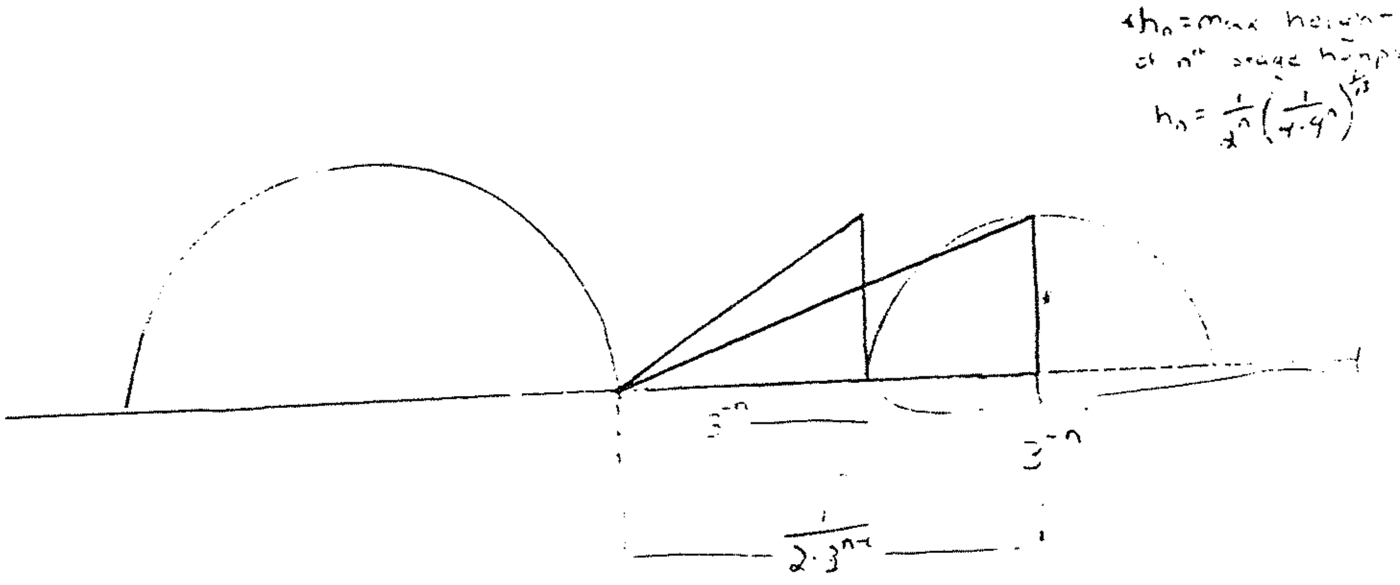
And,

$$\int_{a_{k,n}}^x f' = f(x) - f(a_{k,n}) = f(x) = f(x) - f(0)$$

Therefore,

$$f \in AC\ [0,1]$$

Now, let us consider the differentiability of our function. To do this we will consider several cases. First we will look at the endpoints of removed intervals. To do this we will consider the slope from an endpoint to the nearest n^{th} stage removed interval. It is clear (see figure) that this slope is maximized when the humps maximum height is extended from that hump's nearest endpoint.



Lfor x in the nearest removed n^{th} stage interval, let us consider the ratio

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{f(x)}{x-x_0}.$$

For the triangle with a shallower slope, with x at the midpoint,

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{f(x)}{x-x_0}=\frac{\frac{1}{2^n}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}}}{\frac{1}{2*3^{n-1}}}=\frac{3^{n-1}}{2^{n-1}}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}}.$$

We will now consider the triangle with the steeper slope, and label the ratio k_n:

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{f(x)}{x-x_0}\leq\frac{\frac{1}{2^n}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}}}{\frac{1}{3^n}}=\frac{3^n}{2^n}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}}=k_n.$$

It is clear that

$$\frac{f(x)}{x-x_0} \leq k_n$$

The question now is what happens to k_n as $n \to \infty$.

$$k_n = \frac{3^n}{2^n}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}} = 4^{\frac{-1}{\beta}}\left(\frac{3}{2*9^{\frac{1}{\beta}}}\right)^n$$

Thus, $k_n \to 0$ whenever $\frac{3}{2*9^{\frac{1}{\beta}}} < 1$, that is, whenever $\beta < \frac{\ln 9}{\ln 1.5}$.

This means that f is not differentiable at the endpoints for $1 < \beta < \frac{\ln 9}{\ln 1.5}$ since the derivative from inside the hump does not exist (it is $-\infty$ from the left), but the derivative from outside the hump is zero. Thus, the one sided derivatives do not agree.

Consider $\beta > \frac{\ln 9}{\ln 1.5}$. Then, for all $\delta > 0$,

$$\sup_{x_0<x<x_0+\delta} \left|\frac{f(x)}{x-x_0}\right| = \infty$$

Thus, f is not differentiable at the endpoints for $\beta > \frac{\ln 9}{\ln 1.5}$.

When $\beta = \frac{\ln 9}{\ln 1.5}$ then,

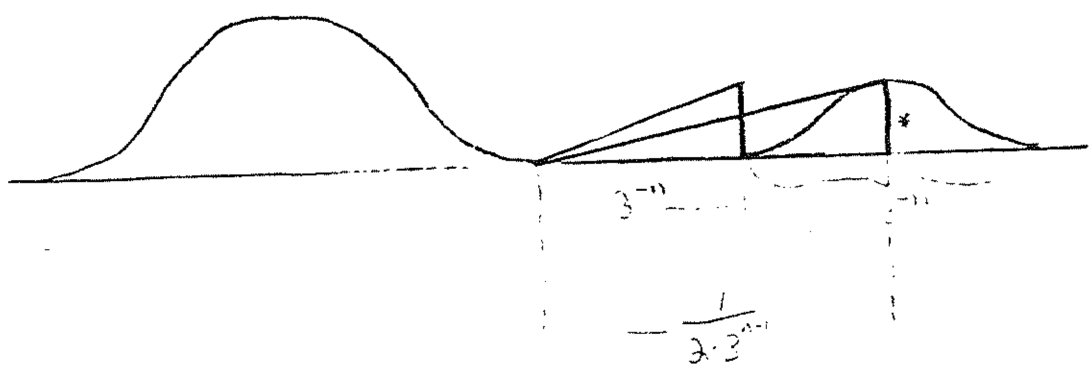
$$\sup_{x_0<x<x_0+\delta} \left|\frac{f(x)}{x-x_0}\right| \geq \frac{2}{3}4^{-\frac{1}{\beta}}.$$

f is not differentiable at the endpoints for any $\beta > 1$

Our goal now will be to show that f^α is differentiable at the endpoints for $\alpha > \beta$. Our function, f^α is defined as :

$$f^\alpha|_{[c,d]}(x) = \left(\frac{1}{2^n}\right)^\alpha [(x-c)(d-x)]^{\frac{\alpha}{\beta}}$$

This function is clearly differentiable from inside each hump, and the one-sided derivative at endpoints is 0. It is necessary to show that the derivative from the outside of each hump exists, and equals 0.



We again consider the ratio of

$$\frac{f(x)}{x - x_0}$$

For the shallower slope we have:

$$\frac{3^{n-1}}{2^{\alpha n-1}}\left(\frac{1}{4 * 9^n}\right)^{\frac{\alpha}{\beta}}$$

And for the steeper slope:

$$\frac{3^n}{2^{\alpha n}}\left(\frac{1}{4 * 9^n}\right)^{\frac{\alpha}{\beta}} = k_n$$

Now, we need $k_n \to 0$ as $n \to \infty$.

$$k_n = 4^{\frac{-\alpha}{\beta}}\left(\frac{3}{2^\alpha * 9^{\frac{\alpha}{\beta}}}\right)^n$$

This goes to zero when

$$\frac{3}{2^\alpha * 9^{\frac{\alpha}{3}}} < 1,$$

which is whenever $3 < 2^\alpha * 9^{\frac{\alpha}{3}}$. This is always true for $\alpha > \beta$, since $2^\alpha > 1$, and $9^{\frac{\alpha}{3}} > 9$

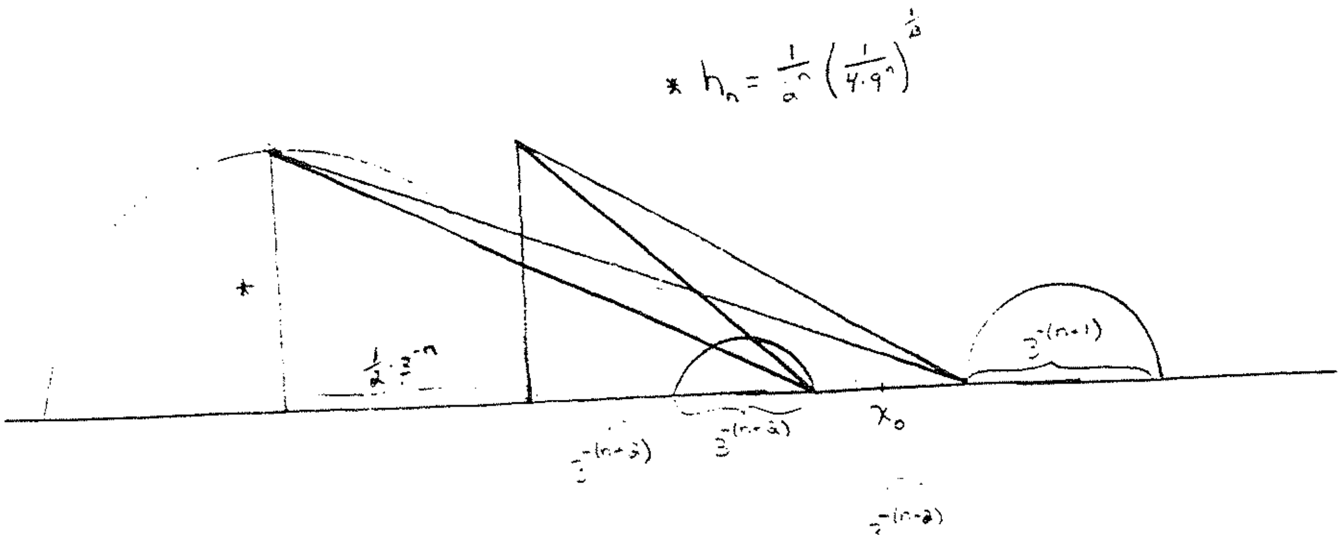
This demonstrates that f^α is differentiable at the endpoints for all $\alpha > 3$.

We have established that f is not differentiable at the endpoints of each removed interval, yet f^α is differentiable at these points. Endpoints, however, form only a countable subset of the Cantor set. It is therefore required that we consider the points of the Cantor set that are not endpoints of a removed interval. To do this we will employ the ternary representation mentioned earlier. We recall that elements of the Cantor set can be expressed in a ternary representation, consisting of only 0's and 2's.

We can imagine three cases to consider, aside from endpoints which have already been dealt with. The first is that of a representation which is eventually $\overline{20}$. In this case, the finite ternary expansions for the point consecutively stays at the right endpoint of a removed interval for one stage and jump to the next such endpoint at the next stage, and this process is carried on infinitely. This will be our "moderate" case, for reasons which will become clear.

The next two cases are quite similar. The one results when the ternary representation eventually has an infinite number of times when consecutive 0's appear. Similarly, the third case is that of a ternary representation which has infinitely many times when consecutive two's appear.

First, let us consider x_0 having a ternary representation which is eventually $\overline{02}$. It is not hard to see that x_0 is contained in some region between humps for any given stage of the Cantor set construction. The difference quotient at x_0 is bounded above by a "steep" slope and whenever x is the midpoint of the nearest removed interval at stage n, the difference quotient is bounded below by a shallow slope. slope goes to 0 as $n \to \infty$, then the ratio for x_0 will be forced to zero.



Here,

$$|\frac{f(x)}{x-x_0}| \leq \frac{3^{n+2}}{2^{n+1}}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}}.$$

Let

$$k_n = \frac{3^{n+2}}{2^{n+1}}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}}.$$

Then f will be differentiable at the point x_0 if $k_n \to 0$, that is, whenever $\frac{3}{2*9^{\frac{1}{\beta}}} < 1$, namely, whenever $\beta < \frac{\ln 9}{\ln 1.5}$.

On the other hand, since

$$|\frac{f(x)}{x-x_0}| \geq \frac{3^{n+1}}{5*2^{n-1}}\left(\frac{1}{4*9^n}\right)^{\frac{1}{\beta}}$$

whenever x is the midpoint of the nearest interval removed at stage n, f is not differentiable at x_0 whenever $\beta > \frac{\ln 9}{\ln 1.5}$.

While the above arguments have only considered slopes to the left, the argument is symmetric if we consider slopes to the right.

Other points $x_0 \in C\backslash E$. can have, as we have mentioned, one of three possible ternary representations: (1)the first we have just seen , and eventually becomes $0\bar{2}$, (2)the second possibility is that the expansion has infinitely many times when consecutive $0's$ appear, and (3) the third possibility is that there are infinitely many times when consecutive $2's$ appear.

Points x_0 which are of type 2 or 3 are necessarily points of nondifferentiability, whenever points of type 1 are points of nondifferentiability. This is because points of type (1) stay between humps, thus maintaining a moderate ratio, while points of type 2 and 3, become much closer to humps, forcing much steeper slopes. These slopes will always be above the slope for functions of type 1, and thus will necessarily cause nondifferentiability, whenever the more mild slope causes nondifferentiability. Thus, we will consider this worst case,such that we have nondifferentiability for all three cases, for our investigation of differentiability at powers of f.

It may not be obvious how points $x_0 \in C\backslash E$, will behave for powers of f. Our contention is that f will be differentiable everywhere for $\alpha > \beta$.

Pick $x_0 \in C$ and a neighborhood (c,d) containing x_0. Without loss of generality, assume (c,d) contains only whole humps of f. Let n be the smallest integer for which (c,d) contains an interval removed at the n^{th} stage, notationally, $E_{k,n} = (a_{k,n}, b_{k,n})$.

Again, without loss of generality, assume x_0 is to the right of this hump.
Consider:

$$\frac{f^\alpha(x) - f^\alpha(x_0)}{x - x_0}$$

for some $x \in E_{k,n}$.
Since $f^\alpha(x_0) = 0$

$$|\frac{f^\alpha(x) - f^\alpha(x_0)}{x - x_0}| = |\frac{f^\alpha(x)}{x - x_0}|$$

Since x_0 is outside of the interval

$$|\frac{f^\alpha(x)}{x - x_0}| \leq |\frac{f^\alpha(x) - f^\alpha(b_{k,n})}{x - b_{k,n}}|$$

since $f^\alpha(b_{k,n}) = 0$

Now , by an application of the Mean Value Theorem, there is a number $\xi \in (x, b_{k,n})$ such that

$$(f^\alpha)'(\xi) = \frac{f^\alpha(b_{k,n}) - f^\alpha(x)}{b_{k,n} - x}$$

By direct calculation, it is easy to see that

$$\max\{|(f^\alpha)'(\xi)| : a_{k,n} \le \xi \le b_{k,n}\} \to 0$$

as $n \to \infty$.

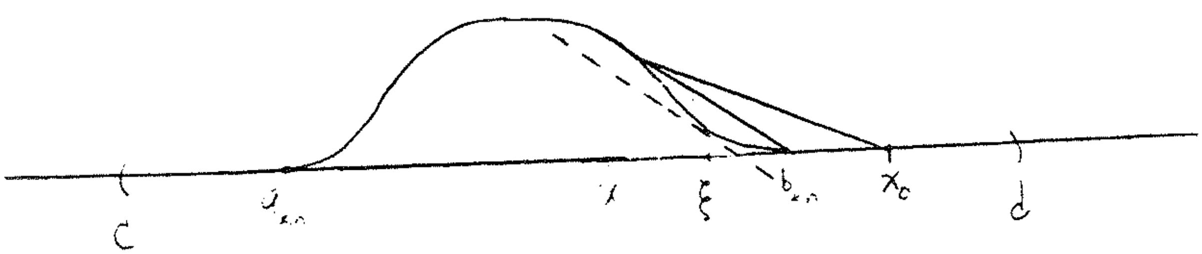
Thus, the ratio $|\frac{f^\alpha(x) - f^\alpha(b_{k,n})}{x - b_{k,n}}|$, represents the absolute value of the slope of f^α at a point ξ, and since these slopes approach 0 as $n \to \infty$, we know that for $\epsilon > 0$ there is some natural number N such that

$$|\frac{f^\alpha(x) - f^\alpha(x_0)}{x - x_0}| \le |\frac{f^\alpha(x) - f^\alpha(b_{k,n})}{x - b_{k,n}}| < \epsilon$$

whenever $n > N$.

This demonstrates that f^α is differentiable everywhere for $\alpha > \beta$

We conclude this example by summarizing our results. We have found that f is not differentiable at the endpoints of removed intervals, but indeed f^α exhibits the desired smoothing effects. Also, for certain choices of β, f is not differentiable at any points of $C \backslash E$. In this scenario, however, for $\alpha > \beta$, f^α again has the desired smoothing effects, and all is well!



Cantor-like function

When studying the Cantor set, it is not unusual to encounter a Cantor-like set, which bears the same properties as the Cantor set, but has positive measure. Such a set is defined by removing less than $\frac{1}{3}$ at each stage.

Let us construct such a set, D, that is perfect, nowhere dense, but such that $\mu(D) = \frac{1}{2}$

Begin by constructing sets F_n, as follows:

$$-F_1 \textit{ is the middle } \frac{1}{4} \textit{ of } [0,1]$$

$$-F_2 \textit{ is the union of the middle open intervals of length } \frac{1}{16} \textit{ in the remaining intervals}$$

$$-F_3 \textit{ is the union of the middle open intervals of length } \frac{1}{64} \textit{in the remaining intervals}$$

•

•

•

We now define,

$$D = [0,1] \setminus \cup_{n=1}^{\infty} F_n \textit{ and } F = \cup_{n=1}^{\infty} F_n.$$

We now calculate the Lebesgue measure of F_n

$$\mu(\cup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n) = \sum_{n=1}^{\infty} \ell(F_n)$$

since each F_n is an interval. Calculating this gives

$$\mu(\cup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Then, calculating the Lebesgue measure of D :

$$\mu(D) = \mu([0,1]) - \mu(\cup_{n=1}^{\infty} F_n) = 1 - \frac{1}{2} = \frac{1}{2}$$

We will simply state as a fact that D is perfect and nowhere dense.

We will now construct a function similar to our Cantor function, but defined using D.

We want $f : [0,1] \to \Re$ such that f has degree β and $F' = f^{-1}\{(0)\}$ is uncountable with $\mu(F') = \frac{1}{2}$.

Let $F' = D$ denote the modified Cantor set. Let each $x \in F'$ be a point where $f(x) = 0$.

Now, as before, let $c < d$ denote the endpoints of some removed interval, and define

$$f|_{[c,d]}(x) = [(x-c)(d-x)]^{\frac{1}{\beta}}$$

As before, for each removed interval $(a_{k,n}, b_{k,n})$, $f \in AC[a_{k,n}, b_{k,n}]$, f is differentiable on $(a_{k,n}, b_{k,n})$, f' is continuous on this interval, and the Fundamental Theorem of Calculus holds on $[a_{k,n}, b_{k,n}]$.

Consider the removed interval (c, d) .

$$f'|_{[c,d]}(x) = 0 \ when \ x = \frac{c+d}{2}$$

This shows us that, as was to be expected, the maximum height of each hump is attained at its midpoint, as was the case with our original Cantor function.

At this point, the height of the hump is given as:

$$f(\frac{c+d}{2}) = 4^{\frac{-1}{\beta}}[(d-c)^2]^{\frac{1}{\beta}}$$

Again we wish to consider the variation of f, so we look at

$$\int_c^d |f'| = T_c^d(f) = 2\left[\frac{(d-c)^2}{4}\right]^{\frac{1}{\beta}}$$

$$\int_F |f'| = \sum \int_{F_i} |f'| = \sum T_{a_{k,n}}^{b_{k,n}}(f) = 4^{-1/\beta} \sum_{n=1}^{\infty} \left[\frac{2}{4^{\frac{2}{\beta}}}\right]^n$$

Then, we will have $f \in BV[0,1]$ when this converges, and the series converges when:

$$\frac{2}{4^{\frac{2}{\beta}}} < 1, \ or$$

$$\beta < 4.$$

As in the example above, absolute continuity also follows, this time whenever $\beta < 4$. This means that for $\beta < 4$, f is differentiable a.e., since every absolutely continuous function is differentiable, except on a set of measure 0.

Again, it seems natural to scale the function so that it is absolutely continuous regardless of β. What scaling is needed to do this?

Suppose we consider a scaling factor of $\left(\frac{1}{\gamma}\right)^n$ on intervals removed at stage n.

$$f|_{[c,d]}(x) = \left(\frac{1}{\gamma}\right)^n [(x-c)(d-x)]^{\frac{1}{\beta}}$$

Now, considering

$$\int_E |f'| = \sum \int_{E_i} |f'| = \sum T_{a_{k,n}}^{b_{k,n}}(f) = \sum_{n=1}^{\infty} \left(\frac{2}{\gamma}\right)^n \left(\frac{1}{4^{2n+1}}\right)^{\frac{1}{\beta}} = \frac{1}{4^{\frac{1}{\beta}}} \sum_{n=1}^{\infty} \left(\frac{2}{16^{\frac{1}{\beta}}\gamma}\right)^n$$

which converges for $\frac{2}{16^{\frac{1}{\beta}}\gamma} < 1$. Hence, we need $\gamma < \frac{2}{16^{\frac{1}{\beta}}}$ for all $\beta > 0$.

We need,

$$\gamma \geq \sup \frac{2}{16^{\frac{1}{\beta}}} = 2$$

Therefore, we need a scaling factor of at least $\frac{1}{2^n}$ in order to get absolute continuity for any given β

Again, we first look at endpoints of our removed intervals. The examination is modeled on that for our original Cantor function. We first calculate our k_n which is the ratio from the endpoint in question to the maximum height of the next hump, extended from its closest endpoint.

$$k_n = \frac{\frac{1}{\gamma^n}\frac{1}{4^{\frac{1}{\beta}}}\left(\frac{1}{16^{\frac{n}{\beta}}}\right)}{\frac{1}{2^{n+1}}\left(1 - \sum_{k=1}^{n+1} \frac{2^{k-1}}{4^k}\right)}$$

This reduces, when we plug in the appropriate γ to

$$k_n = \frac{2 \cdot 16^{-n/\beta}}{4^{\frac{1}{\beta}}\left(1 - \sum_{k=1}^{n+1} \frac{2^{k-1}}{4^k}\right)}$$

and as $n \to \infty$. $k_n \to 0$ but the derivative from inside each hump fails to exist at endpoints for $\beta > 1$ being $-\infty$ or ∞. so f fails to be differentiable at the endpoints.

A similar evaluation for f^α results in the following ratio (reduced and γ already substituted in):

$$k_n = \frac{4 \cdot 16^{-n\alpha/\beta}}{2^{(\alpha-1)n}4^{\frac{\alpha}{\beta}}\left(1 - \sum_{k=1}^{n+1} \frac{2^{k-1}}{4^k}\right)}$$

and for this ratio $k_n \to 0$ as $n \to \infty$. For $\alpha > \beta$, the derivative in side each hump at endpoints is also 0, showing that f^α is differentiable for $\alpha > \beta$.

We now consider, the points of $[0,1]$ that have a "ternary expansion" which is eventually $\overline{20}$. We are not using ternary form, but rather the idea associated with it of where points go, for example, the idea of 2 being a jump, and 0 a stay. On the modified Cantor set, these will be points that stay between humps. They do not approach a given hump. A calculation of the ratio $\frac{f(x)}{x-x_0}$ for our unscaled function, yields

$$k_n = \frac{4^{1-1/\beta} \cdot 2^{n+1}}{20 + 4 \cdot 2^n} \left(\frac{2}{16^{1/\beta}} \right)^n.$$

In this case, $k_n \to 0$ as $n \to \infty$ as long as $\beta > 4$ (as before, for absolute continuity), showing that f is differentiable at these $\overline{20}$ points in this case.

Here, we wish to scale the function so that it is not differentiable at these $\overline{20}$ points. If the function is not differentiable at these points, then it will not be differentiable at other points of $D \setminus \cup_{n=1}^{\infty} \overline{F}$ which yield steeper slopes. Since these slopes are bounded below by a $\overline{20}$ slope, they will necessarily not approach zero when the more moderate slope do not approach zero. Again we will consider this worst case.

Consider the shallower slope, bounding our $\bar{20}$. We will scale the maximum height by a factor of $\frac{1}{\gamma}$. If this scaled height does not go to zero, we will have nondifferentiability for the $\bar{02}$ case, and consequently for all points of D.

Let k_n represent the slope for the scaled shallow height:

$$k_n = \frac{\frac{1}{\gamma^n} \frac{1}{4^{\frac{1}{\beta}}} \left(\frac{1}{16^{\frac{1}{\beta}}} \right)^n}{\frac{1}{2^{2n+1}} + \frac{1}{2^{n+2}} + \frac{3}{2^{2n+4}}} = \frac{4}{4^{\frac{1}{\beta}}(1 + \frac{1}{2^{n-1}} + \frac{3}{2^{n+2}})} \left(\frac{2}{\gamma 16^{\frac{1}{\beta}}} \right)^n$$

This means that the function will be nondifferentiable at these points for $\frac{2}{\gamma 16^{\frac{1}{\beta}}} \geq 1$.

For a given, β choose

$$\gamma = \frac{2}{16^{\frac{1}{\beta}}}.$$

Now that we have scaled the function, we need to verify its continuity. To do so, we must be sure that the heights of the humps go to 0. Let h_n be the height of some hump.

$$\frac{1}{\gamma^n} h_n = \left(\frac{16^{\frac{1}{\beta}}}{2} \right)^n \frac{1}{4^{\frac{1}{\beta}}} \left(\frac{1}{16^{\frac{1}{\beta}}} \right)^n = \frac{1}{4^{\frac{1}{\beta}}} \frac{1}{2^n} \to 0 \quad as\ n \to \infty.$$

Then, for the appropriate γ we have f continuous and nondifferentiable at all points of $D\backslash F$, regardless of their location.

Does raising this function have the effect that we are hoping for, that is , does f^α become differentiable at these points for some α?

We have:

$$f^\alpha|_{[c,d]}(x) = \frac{1}{\gamma^{n\alpha}}[(x-c)(d-x)]^{\frac{\alpha}{\beta}}$$

where γ is the appropriate scaling factor for the chosen β.

We will consider the $\bar{2}0$ case for differentiability. Once, more we consider, the steeper slope, since if this goes to 0, the slope in question will be forced to 0. Note that differentiability, for the more moderate case, does not imply differentiability for all points of $D\backslash\overline{F}$.

The ratio to be considered for the steeper slope is:

$$\frac{\frac{1}{\gamma^{\alpha n}}\left(\frac{1}{4^{2n+1}}\right)^{\frac{\alpha}{\beta}}}{\frac{1}{4^{n+2}}+\frac{1}{2^{n+2}}(1-\sum_{k=1}^{n+2}\frac{2^{k-1}}{4^k})} = \frac{16}{4^{\frac{\alpha}{\beta}}\left(\frac{1}{2^n}+4(1-\sum_{k=1}^{n+2}\frac{2^{k-1}}{4^k})\right)}\left(\frac{2}{2^\alpha}\right)^n$$

Thus $k_n \to 0$ *as* $n \to \infty$ despite β as long as $\alpha > 1$ So, f^α is differentiable at all points with the representation $\overline{20}$. The exact same argument used to prove differentiability at **all** points of $C\backslash\overline{F}$ for f^α in the Cantor set example can here be applied equally effectively to the Cantor-like example, to show that f^α for $\alpha > \beta$ is differentiable for every point of $D\backslash F$.

To summarize, we again have encountered the smoothing effects of raising f to a power. Here, we have examined the effects on a set of positive measure, whereas the Cantor function previously explored examined a set of measure zero.

INTEGRATION

The Fundamental Theorem of Calculus proposes an interesting question in the study of degrees of differentiability. For example, given a function f, which is differentiable, we have

$$(f^\alpha)' = \alpha f^{\alpha-1} * f'.$$

If we know that f is not differentiable, in rearranging the previous expression, we could consider

$$\frac{(f^\alpha)'}{\alpha f^{\alpha-1}}$$

which would be f' if it existed.

The question now is whether or not

$$\frac{(f^\alpha)'}{\alpha f^{\alpha-1}}$$

works like a derivative.

One interesting question to consider is what happens when we take the definite integral over $[a, b]$ of $\frac{(f^\alpha)'}{\alpha f^{\alpha-1}}$.

First, let's consider a simple example:

Recall:

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{otherwise.} \end{cases}$$

and

$$f^\alpha(x) = |x|^\alpha = \begin{cases} x^\alpha, & \text{if } x \geq 0; \\ (-x)^\alpha, & \text{otherwise.} \end{cases}$$

If we take the definite integral of $\frac{(f^\alpha)'}{\alpha f^{\alpha-1}}$ over an interval $[a, b]$ we get three cases:

(1)$0 \leq a < b$

$$\int_a^b \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} = b - a = f(b) - f(a) \quad since \quad a, b > 0$$

(2)$a < b \leq 0$

$$\int_a^b \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} = -b + a = f(b) - f(a) \quad since \quad a, b \leq 0$$

(3)$a \leq 0$, $b > 0$

$$\int_a^b \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} = a + b = f(b) - f(a) \quad since\ a \leq 0,\ b > 0$$

It comes as no suprise, that somewhat lofty restrictions must be placed on our function in order for the FTC to hold. In fact, the restrictions, are the typical restrictions required for the FTC.

Lebesgue Case:

Claim. Suppose $f : [a, b] \to \Re$ is absolutely continuous. Let $E' = \{x \in [a, b] | f(x) = 0\}$ be a set of measure zero, and let f be differentiable except at $x \in E'$, with f^α differentiable everywhere for some $\alpha > \beta$. Then,

$$\int_a^b \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} d\mu = \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} d\mu = f(b) - f(a)$$

where, $a = a_0$, $b = a_{N+1}$. *and all other* $a_i = x_i$

proof

Let $E = \{x \in [a, b] | f(x) > 0\} = \cup_{n=1}^{N+1}(a_{n-1}, a_{an})$. Then for $[a, b] \backslash E = E'$, $mu(E') = 0$.

Now, pick any $x_i, x_{i+1} \in E'$ and consider $[x_i, x_{i+1}]$. Then f is differentiable on $(x_i, x_i + 1)$, but not at x_i, x_{i+1}; while f^α is differentiable on $[x_i, x_{i+1}]$ for $\alpha > \beta$.

Define a sequence of functions $\{g_n\}$:

$$g_n(x) = \frac{(f^\alpha)'(x)}{\alpha f^{\alpha-1}(x)} \chi_{[x_i + \frac{1}{n}, x_{i+1} - \frac{1}{n}]}$$

$$g(x) = \frac{(f^\alpha)'(x)}{\alpha f^{\alpha-1}(x)} \chi_{[x_i, x_{i+1}]}$$

$|g_n(x)| \leq g(x)$ Since the limit of a sequence of measurable functions is itself measurable, and

$$g_n \to g$$

by the Lebesgue Dominated Convergence Theorem, we know that g(x) is both measurable, and integrable.

Thus,

$$\int_{x_i + \frac{1}{n}}^{x_{i+1} - \frac{1}{n}} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} = \int_{x_i + \frac{1}{n}}^{x_{i+1} - \frac{1}{n}} f'$$

is integrable.

And, because we know that f is absolutely continuous:

$$\int_{x_i + \frac{1}{n}}^{x_{i+1} - \frac{1}{n}} f' = f(x_{i+1} - \frac{1}{n}) - f(x_i + \frac{1}{n}) \to f(x_{i+1}) - f(x_i) = 0$$

We know that f is absolutely continuous on $[a,b]$ so it is also absolutely continuous on $[a,x_1]$ and $[x_N,b]$. As above,

$$\int_a^{x_1} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} \quad \textit{exists, and} \quad \int_a^{x_1} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} = f(x_1) - f(a) = 0 - f(a)$$

Similarly,

$$\int_{x_N}^{b} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} \quad \textit{exists, and} \quad \int_{x_N}^{b} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} = f(b) - f(x_N) = f(b) - 0$$

Thus, combining the integrals over each subsection,

$$\int_a^b \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} d\mu = \int_a^{x_1} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} + \sum_{i=1}^{N-1} \int_{x_i}^{x_i+1} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} + \int_{x_N}^{b} \frac{(f^\alpha)'}{\alpha f^{\alpha-1}} = f(b) - f(a)$$

We see that absolute continuity is a necessary restriction on f in order to have the Fundamental Theorem of Calculus hold in the Lebesgue Case.

Note This argument works as long as E' is finite. Otherwise, under the hypothesis of absolute continuity, and E' being both the zero set and set of nondifferentiable points of f, $\frac{(f^\alpha)'}{\alpha f^{\alpha-1}}$ and f' are identical; that is, their domains are the same, and on that common set (E), they agree point by point, so the conclusion then follows from absolute continuity of f.

CONCLUSION

In studying degrees of differentiation, many interesting conclusions can be drawn. Although, we attempted to develop the properties and characteristics of functions possessing a degree of differentiability, many questions still remain. Hopefully, this paper has served to answer some questions surrounding the topic, as well as to provide opportunities for future study.

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