Observing KAM Structure in a Forced Pendulum

Jason Osborne[1]

August 3, 1999

[1]This research was conducted during an NSF funded Research Experience for Undergraduates program at Northern Arizona University under the guidance of Dr. Jim Swift.

The minimum criteria for a system to show chaos is be a system of two ordinary differential equations with some sort of non-linear time dependent driving term. A pendulum attached to a sinusoidally moving ceiling meets these minimum conditions. As introduction, we use the next three sections to examine a spring-block system attached to a sinusoidally moving ceiling. The spring has natural length ℓ and spring constant k, the block has a mass m, and the wall moves sinusoidally according to $\epsilon\cos(\omega t)$ where ϵ is the amplitude of the oscillation and ω is the frequency. In section 1 the problem is studied from an inertial or laboratory reference frame while in section 2 a non-inertial or accelerating refrence refrence frame is used. In section 3 the inertial and non-inertial reference frame are linked together. In section 4 we utilize Lagrange's equation of motion to describe the motion of the spring. In section 5, we examine KAM structure in a forced pendulum.

1 Laboratory Reference Frame

In this section we consider our refrence frame to be the laboratory. For the time being we will say that all measurements are being made from the floor, see Figure 1. Thus we denote $\vec{x}$ to be the vector from the floor to the block and $\vec{x_c}$ to be the vector from the floor to the moving ceiling. We consider the ceiling to be a distance d above the floor before it begins oscillating. That is,

$$\vec{x_c} = (d + \epsilon\cos(\omega t))\hat{k}$$

or

$$x_c = d + \epsilon\cos(\omega t).$$

The total force on the block can be expressed as

$$F = -mg + k(x_c - x - \ell). \tag{1}$$

If we now translate the origin some constant, x^*, vertically then the distance between the block and this new origin can be expressed as

$$\begin{aligned} \vec{\bar{x}} &= \vec{x} - \vec{x^*} \\ (\bar{x})\hat{k} &= (x)\hat{k} - (x^*)\hat{k} \end{aligned}$$

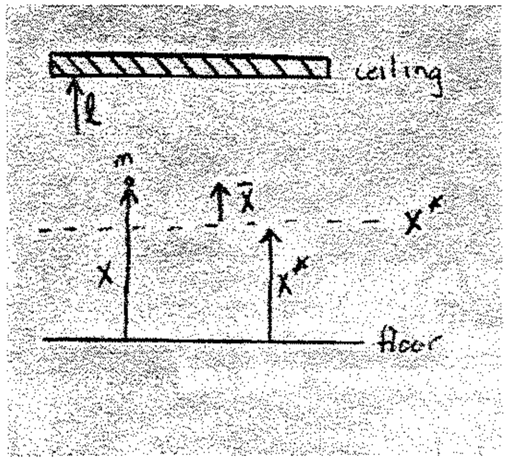


Figure 1: Inertial Refrence Frame with $\epsilon = 0$

or

$$\bar{x} = x - x^{\star} \tag{2}$$

Solving eq. (2) for x and substituting into eq. (1) yields

$$F = -k\bar{x} + k\epsilon\cos(\omega t) + (kd - kx^{\star} - k\ell - mg). \tag{3}$$

If we choose $x^{\star}$ to be

$$x^{\star} = d - l - \frac{mg}{k}$$

then eq. (3) reduces to the equation of motion of the spring-mass system in a laboratory refrence frame,

$$m\ddot{\bar{x}} = -k\bar{x} + k\epsilon\cos(\omega t). \tag{4}$$

For $\epsilon = 0$, that is an unmoving ceiling, eq.(4) obviously reduces to the expression for Hooke's law

$$F = m\ddot{\bar{x}} = -k\bar{x}. \tag{5}$$

Thus $\bar{x}$ represents the distance from the equilibrium point $x^{\star}$. Since $x^{\star}$ is chosen independent of the motion of the ceiling it will remain an equilibrium point for $\epsilon > 0$.

2 Non-Inertial Reference Frame

In this section we consider our refrence frame to be the ceiling and thus we consider what happens when all distances are measured from there, see

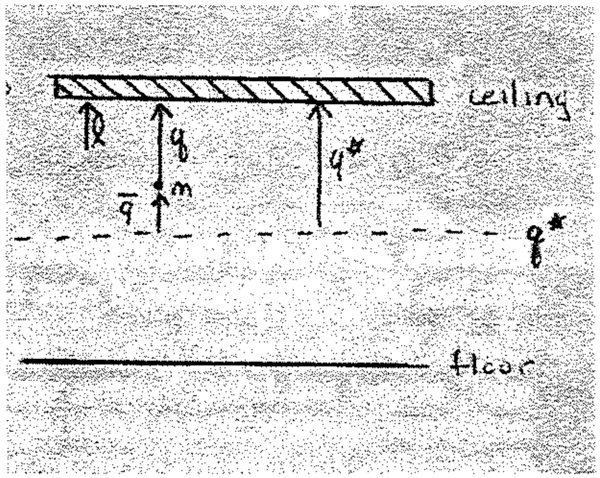


Figure 2: Non-Inertial Refrence Frame, $\epsilon = 0$

Figure 2. We define $\vec{q}$ to be the vector from the mass to the ceiling. We can write the total forces on the mass m as

$$F = mg - k(q - \ell) + f(\epsilon \cos(\omega t)) \tag{6}$$

where $f(\epsilon \cos(\omega t))$ is an as yet undetermined force caused by the acceleration of the refrence frame. A force of this type is called a *pseudo-force*. As an example of a pseudo-force consider a person standing on scale in a elevator. In this scenario the non-inertial refrence frame is the elevator. As the elevator accelerates in the positive direction (upward) the person on the scale will notice their weight to increase (a pseudo-force in the negative direction). Similarly, as the elevator accelerates in the negative direction (down) the person will notice their weight decrease (a pseudo-force in the positive direction). In both instances the pseudo-forces would not have been present had the elevator not been accelerating.

We would now like to express the eq. (6) not in terms of displacement from the natural length of the spring but rather in terms of displacment from some equilibrium point $q^\star$. If we let $\epsilon = 0$ then eq. (6) reduces to

$$F = mg - kq + k\ell. \tag{7}$$

At the point of equilibrium, $q = q^\star$, the forces balance and thus $F = 0$. Hence $q^\star$ can be expressed as

$$\begin{aligned} mg - kq^\star + k\ell &= 0 \\ \ell + \frac{mg}{k} &= q^\star. \end{aligned}$$

Defining $\vec{\bar{q}}$ to be the difference of vectors $\vec{q^\star}$ and $\vec{q}$, see Figure 2, yields the expression

$$\begin{aligned} \vec{q} &= \vec{q^\star} - \vec{\bar{q}} \\ (q)\hat{k} &= (q^\star)\hat{k} - (\bar{q})\hat{k} \end{aligned}$$

or

$$q = q^\star - \bar{q}. \tag{8}$$

It is important to realize that when the ceiling is set in motion the *equilibrium* point will vary sinusoidally. If we substitute eq. (8) into eq. (7) to get

$$\begin{aligned} F &= mg - k(\bar{q} + q^\star - l) \\ &= -k\bar{q} + (mg - kq^\star + kl). \end{aligned}$$

and recognize the term in parentheses as the force at the equilibrium point we get

$$F = -k\bar{q}.$$

If we now allow the ceiling to move and thus reintroduce the pseudo-force the equation of motion in a non-inertial refrence frame is given by

$$F = m\ddot{\bar{q}} = -k\bar{q} + f(\epsilon\cos(\omega t)). \tag{9}$$

The pseudo-force can be determined by examining the relationship between the inertial and non-inertial refrence frame.

3 Linking the Inertial and the Non-Inertial Refrence Frames

In the last two sections we found both $x^\star$ and $q^\star$ by considering $\epsilon = 0$ which lead to the conclusion that both $x^\star$ and $q^\star$ represent the equilibrium point.

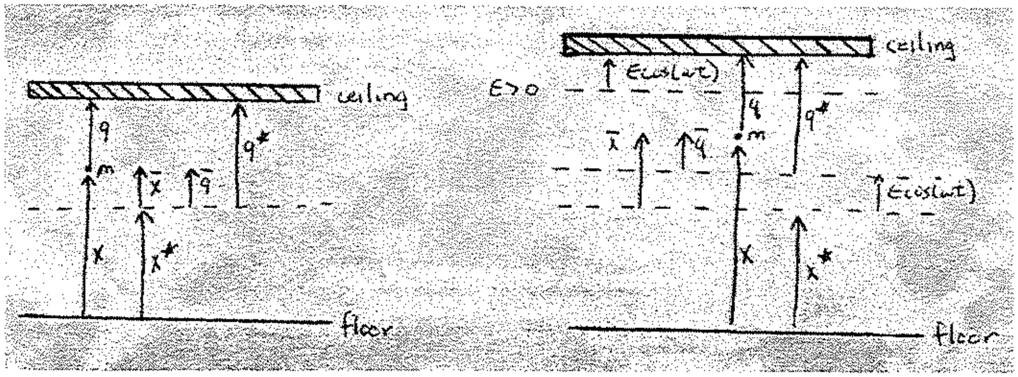


Figure 3: a. (left) Inertial and Non-Inertial for $\epsilon = 0$ b. (right) Inertial and Non-Inertial for $\epsilon > 0$

Using this knowledge and the relations given in eq. (2) and eq. (7) leads us to Figure 3a. and to the conclussion that $\bar{x} = \bar{q}$. We would now like to consider what happens when $\epsilon > 0$. Making use of the knowledge that the *equilibrium* point q^* moves as the ceiling moves, x^* is constant in time and the relations given in eq. (2) and eq. (8) lead us to Figure 3b.

From this figure it is obvious that

$$\begin{aligned} (\bar{x})\hat{k} - (\bar{q})\hat{k} &= (\epsilon\cos(\omega t))\hat{k} \\ \bar{x} - \bar{q} &= \epsilon\cos(\omega t) \end{aligned} \tag{10}$$

This equation relates one refrence frame to the other. That is, if we solve eq. (10) for $\bar{x}$ and substitute into eq. (4) to get

$$m(\ddot{\bar{q}} - \epsilon\omega^2\cos(\omega t)) = -k(\bar{q} - \epsilon\cos(\omega t)) - k\epsilon\cos(\omega t). \tag{11}$$

and reduce yields

$$m\ddot{\bar{q}} = -k\bar{q} + m\epsilon\omega^2\cos(\omega t). \tag{12}$$

Equating eq. (9) and eq. (12) we find the pseudo-force term to be

$$f(\epsilon\cos(\omega t)) = m\epsilon\omega^2\cos(\omega t) \tag{13}$$

and thus the equation of motion in a non-inertial refrence frame is given by

$$m\ddot{\bar{q}} = -k\bar{q} + m\epsilon\omega^2\cos(\omega t). \tag{14}$$

4 Lagrangian Approach

In this section we examine the spring-block system from an energy standpoint. We define the Lagrangian to be

$$L = T(\dot{x}) - U(x) \tag{15}$$

where T is the kinetic energy and U is the potential energy. The potential energy is defined to be

$$U(x) = -\int_{x_0}^{x} F(x)dx$$

and the kinetic energy is defined to be

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2.$$

Using the force described in eq. (1) to find the potential energy yields

$$\begin{aligned} U(x) &= -\int_{x_0}^{x}(-mg + k(x_c - x - \ell))dx \\ &= -\left[-mgx|_{x_o}^{x} - \frac{k}{2}(x_c - x - \ell)^2|_{x_o}^{x}\right] \end{aligned} \tag{16}$$

Since we can measure a potential energy from anywhere we choose x_o such that

$$\frac{k}{2}(x_c - x - \ell)^2|_{x_o} = 0$$

Thus,

$$x_0 = x_c - \ell.$$

Now evaluating the integral in eq. (16), with x_o given above yields

$$\begin{aligned} U(x) &= mgx - mg(x_c - \ell) + \frac{k}{2}(x_c - x - \ell)^2 \\ &= -mg(x_c - x - \ell) + \frac{k}{2}(x_c - x - \ell)^2. \end{aligned} \tag{17}$$

Substituting x_c defined in section 1, $\epsilon = 0$, and $x = \bar{x} - x^*$ into eq. (16) yields

$$\begin{aligned} U(x) &= mg\bar{x} - mgd + mgd - mg\ell + mg\ell - \frac{m^2g^2}{k} + \frac{k}{2}\left(d - d + \ell - \ell - \bar{x}^2 + \frac{m^2g^2}{k^2}\right) \\ &= \frac{1}{2}k\bar{x}^2 + mg\bar{x} - mg\bar{x} + \frac{m^2g^2}{2k} - \frac{m^2g^2}{k} \\ &= \frac{1}{2}k\bar{x}^2 - \frac{m^2g^2}{2k}. \end{aligned} \tag{18}$$

Since $\frac{m^2g^2}{2k}$ is a constant the potential energy given in eq. (18) will give the same motion as a potential of

$$U(x) = \frac{1}{2}k\bar{x}^2. \tag{19}$$

If instead we let $\epsilon > 0$ then eq. (17), through use of MAPLE, reduces to

$$U(x) = \frac{1}{2}k(\bar{x} - \epsilon\cos(\omega t))^2 - \frac{m^2g^2}{2k}. \tag{20}$$

were we once again disregard the constant to obtain a potential energy of

$$U(x) = \frac{1}{2}k(\bar{x} - \epsilon\cos(\omega t))^2. \tag{21}$$

Notice that we could have obtained eq. (21) by directly integrating the force given in eq. (4)

$$U(\bar{x}) = \int_{x_0}^{\bar{x}} (k\bar{x} - k\epsilon\cos(\omega t))d\bar{x} = \frac{1}{2}k(\bar{x} - \epsilon\cos(\omega t))^2$$

where we define the potential energy at x_o to be zero. Using the potential energy in eq. (21) we can write the Lagrangian as

$$L = \frac{1}{2}m(\dot{\bar{x}})^2 - \frac{1}{2}k(\bar{x} - \epsilon\cos(\omega t))^2. \tag{22}$$

We define the integral

$$J = \int_{\bar{x}_1}^{\bar{x}_2} (T-U)dx = \int_{\bar{x}_1}^{\bar{x}_2} L(\bar{x}, \dot{\bar{x}})d\bar{x} \tag{23}$$

for which we would like to find an extremum. In most physical situations J will be a minimum. Using the calculus of variations, see [3, page 235] for details, it turns out that a necessary condition for J to be minimized is

$$\frac{\partial L}{\partial \bar{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\bar{x}}} = 0. \tag{24}$$

Eq. (24) is called the Lagrange equations of motion. Using the Lagrange equations of motion on eq. (22) yields

$$\begin{aligned} 0 &= -k(\bar{x} - \epsilon\cos(\omega t)) - m\ddot{\bar{x}} \\ m\ddot{\bar{x}} &= -k\bar{x} + \epsilon\cos(\omega t) \end{aligned} \tag{25}$$

which is the equation of motion we found in section 1 using Newton's laws. If we now define $\bar{x}$ and $\dot{\bar{x}}$ in terms of generalized coordinates, as in section 2, to be

$$\begin{aligned} \bar{x} &= \bar{x}(\bar{q}, t) \\ \dot{\bar{x}} &= \dot{\bar{x}}(\bar{q}, \dot{\bar{q}}, t) \end{aligned}$$

the Lagrangian can be expressed as

$$L = T(\bar{q}, \dot{\bar{q}}, t) - U(\bar{q}, t) \tag{26}$$

where Lagrange equations of motion are given by [3, page 242]

$$\frac{\partial L}{\partial \bar{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\bar{q}}} = 0. \tag{27}$$

From the relation given in eq. (10) it follows that

$$\dot{\bar{x}} = \dot{\bar{q}} - \epsilon\omega\sin(\omega t). \tag{28}$$

Substituting eq. (10) and eq. (28) into eq. (22), the Lagrangian for section 2 can be written as

$$L = \frac{1}{2}m(\dot{\bar{q}} - \epsilon\omega\sin(\omega t))^2 - \frac{1}{2}k\bar{q}^2. \quad (29)$$

Using eq. (27) yields

$$\begin{aligned} \frac{\partial L}{\partial \bar{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\bar{q}}} &= -k\bar{q} - m\frac{d}{dt}(\dot{\bar{q}} - \epsilon\omega\sin(\omega t)) = 0 \\ &= -m\ddot{\bar{q}} - k\bar{q} + m\epsilon\omega^2\cos(\omega t) \end{aligned} \quad (30)$$

and thus

$$m\ddot{\bar{q}} = -k\bar{q} + m\epsilon\omega^2\cos(\omega t) \quad (31)$$

which was the equation of motion derived from Newton's laws in section 2.

5 The Very Basics of KAM Theory

The KAM theorem (Komolgorov-Arnold-Moser) proves that under certain conditions there exists an invariant torus, expressed in terms of action-angle variables by ($\mathbf{J}$,θ), when an integrable system with Hamiltonian H($\mathbf{J}$,θ) is slightly perturbed. The invariant torus (curve) is called a KAM curve. In [2, page 159] the conditions are stated under which the KAM theorem holds and can be summarized as:

1. frequency is not constant with increasing amplitude [2, page 162]
2. a sufficiently differentiable driving term (perturbation)
3. initial conditions sufficiently far from resonance (fixed points).

Condition 1 rules out simple harmonic oscillators since, for springs, frequency is given by the expression

$$\omega = \sqrt{\frac{k}{m}}$$

where k is the spring constant and m is the mass and for pendulums

$$\omega = \sqrt{\frac{g}{\ell}}$$

where g is the acceleration of gravity and ℓ is the length of the pendulum. Both expression are obviously independent of amplitude. However, a pendulum attached to a sinusoidally moving ceiling will satisfy condition 1. For this system we get the differential equation

$$m\ell^2\ddot{\theta} = -(mg + m\epsilon\omega^2\cos(\omega t))\sin(\theta) - \nu\dot{\theta}. \tag{32}$$

where we recognize the pseudo-force term from section 2 and $\nu\dot{\theta}$ to be the friction in the system. Also recognize that this differential equation is just the expression for the torque on the mass. In dimensionless form the differential equation can be expressed as

$$\ddot{\theta} = -(1 + a\cos(2\pi\tau))\sin(\theta) - \nu\dot{\theta} \tag{33}$$

where τ is the dimensionless time and a is a constant. The non-linear time dependence guarantees that our frequency will not be constant as amplitude increases.

An interesting property of our differential equation, for $\nu = 0$, is that it is area preserving. Analagous to the way Strogatz defines change in volume we can express the change in area by

$$\dot{A} = \int_A (\nabla \cdot \mathbf{f})dA \tag{34}$$

where $\mathbf{f}$ is the instantaneous velocity of a point in the area [4, page 313]. To find the instantaneous velocities we decompose our second order dimensionless differential equation. This results in

$$\begin{aligned} \dot{\theta} &= \alpha \\ \dot{\alpha} &= -(1 + a\cos(2\pi\tau))\sin(\theta) - \nu\alpha \end{aligned} \tag{35}$$

and thus

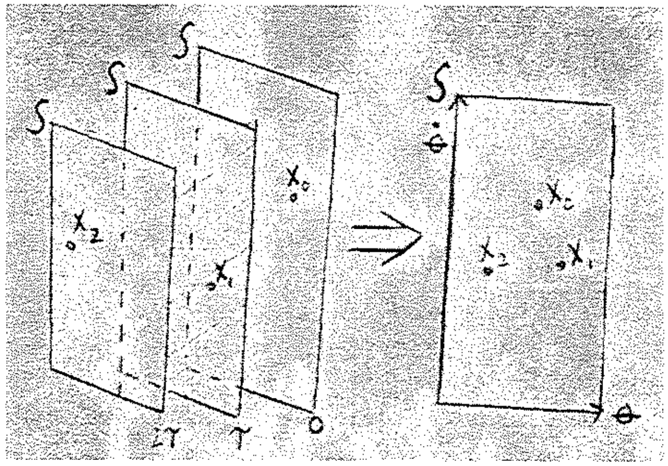


Figure 4: *Stroboscopic* Poincare map that is mod(1) periodic in τ

$$
\begin{aligned}
\nabla \cdot \mathbf{f} &= \frac{\partial}{\partial \theta}(\alpha) + \frac{\partial}{\partial \alpha}(-(1 + a\cos(2\pi\tau))\sin(\theta) - \nu\alpha) \\
&= \begin{cases} 0 & \text{if } \nu = 0 \\ -\nu & \text{if } \nu > 0 \end{cases}
\end{aligned}
$$

which for $\nu = 0$ implies $\dot{A} = 0$ and hence area is preserved and $\nu > 0$ implies area is decreasing. Throughout the rest of the paper we will assume we have no friction and thus $\nu = 0$.

Returning to the KAM theory conditions, condition 2 obviously holds since the driving term is a trigonometric function. Before moving on to condition 3 we need to define a Poincare map [4, page 278]. If we define $\dot{x} = f(x)$ to be some n-dimensional system and we let S be an $n-1$-dimensional surface of section then the Poincare map P is a mapping from S onto itself. That is, follow the trajectory corresponding to initial condition x_o until the trajectory passes through S, call this point x_1. We can then define the Poincare map to be

$$x_{k+1} = P(x_k).$$

We will be most interested in a *stroboscopic* Poincare map. The idea behind this map is that we want to observe the position of the pendulum initially and then observe it one period of the driving term later. Thus our surface of section is, see Figure 3,

$$S = \{(\theta, \dot{\theta}) : \tau = 0 mod(1)\}.$$

We will use this map when we try to find orbits of a particular winding number. For our problem of the forced pendulum, we define the winding number to be

$$\frac{\omega_n}{\omega} = \frac{p}{q}$$

where ω_n is the natural frequency of the pendulum, ω is the frequency of the ceiling. The winding numbers break down into two categories, *commensurate* or $\frac{p}{q}$ rational and *incommensurate* or $\frac{p}{q}$ irrational. To illustrate the concept of rational winding number we choose $p = 1$ and $q = 2$. Physically, this implies that the ceiling can complete two periods of motion in the same time the pendulum completes one period. We then say that the pendulum is period 2 (with respect to the ceiling). Thus winding number $\frac{1}{2}$ and period 2 mean the same thing. Similarily, if $p = 1$ and $q = 1$ then we have a period 1 orbit [1] If $a = 0$, one would then have an infinite number of different phase period 1 orbits in phase space $(\dot{\theta} - \theta)$. This can easily be seen in configuration space $(\tau - \theta)$, see Figure 3.

If we now make $a > 0$ then we are left with only four period 1 orbits in configuration space corresponding to four fixed points in phase space. The banana shaped islands correspond to some rational winding number, see Figures 7 and 8. If, however, we have an orbit that has an incommensurate winding number then the orbit will be densely filled by points. For sufficiently small perturbation and an incommensurate winding number sufficiently far from a fixed point (i.e. not well approximated by a rational), that orbit will be preserved (i.e. not turned into an island) and hence is a KAM curve, see Figure 6. See [2, page 159,167] for details on how far and how small is sufficiently. However, there can be found an a such that an orbit with incommensurate winding number will not give a KAM curve. It turns out that an orbit corresponding to an incommensurate winding number that is not well approximated by a rational will stay around longer than one that is well approximated by a rational [2, page 167]. The potential KAM curve in Figure 4 becomes stochastic at $a = 0.6$. From Figure 6 and 7 we notice that the heteroclinic point, the orbit corresponding to the pendulum approaching standing straight up and down, becomes stochastic as we go from $a = 0$ to $a = 0.08$.

[1] In actuality we have let $q = 0.9$. See the next section for the actual data that was used to generate all of the the orbits in this section.

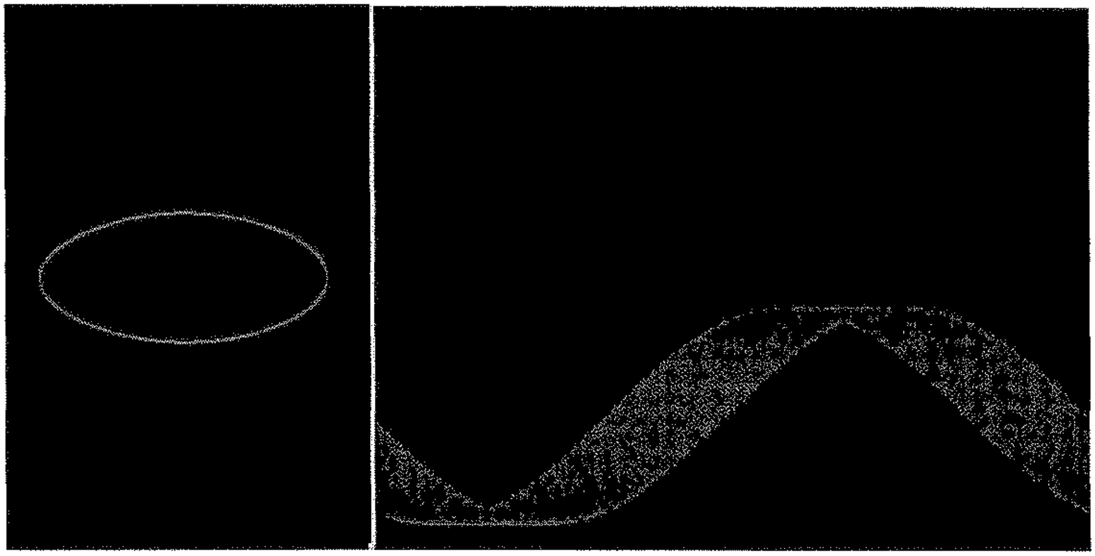


Figure 5: (left) Phase space trajectories for an infinite number of period 1 orbits. (right) Some of the period 1 orbits viewed in configuration space.

To illustrate the idea of having an infinite number of fixed points before a perturbation and an integer number of fixed points after we consider

$$\begin{aligned}\dot{r} &= r(1-r)\\ \dot{\theta} &= \epsilon \sin(2\theta).\end{aligned}$$

Remembering that a fixed point occurs when $\dot{r} = 0$, we find that r has fixed points at $r = 0$ and $r = 1$. For $\dot{\theta} = 0$ and $\epsilon = 0$ we find that for any value of θ there is a fixed point. When $\epsilon = 0$, $r = 0$ and $r = 1$ are still fixed points but there are only four fixed points for θ. These are $\theta = \frac{k\pi}{2}$ where $k = 1, 2, 3, 4$, see Figure 5.

As an example of why we say *potential* KAM curve, we examine $\frac{w_n}{w} = \frac{5}{6}$ which yields a period 6 orbit. This example illustrates a chain of 6 islands, see Figure 9. For these winding numbers between major resonances, in this case between the period 0.5 and period 1, resonances the islands width drastically decreases [2, page 173]. Thus for period 1000 orbits, for example, the islands will be so small that essentially it will look like a KAM curve.

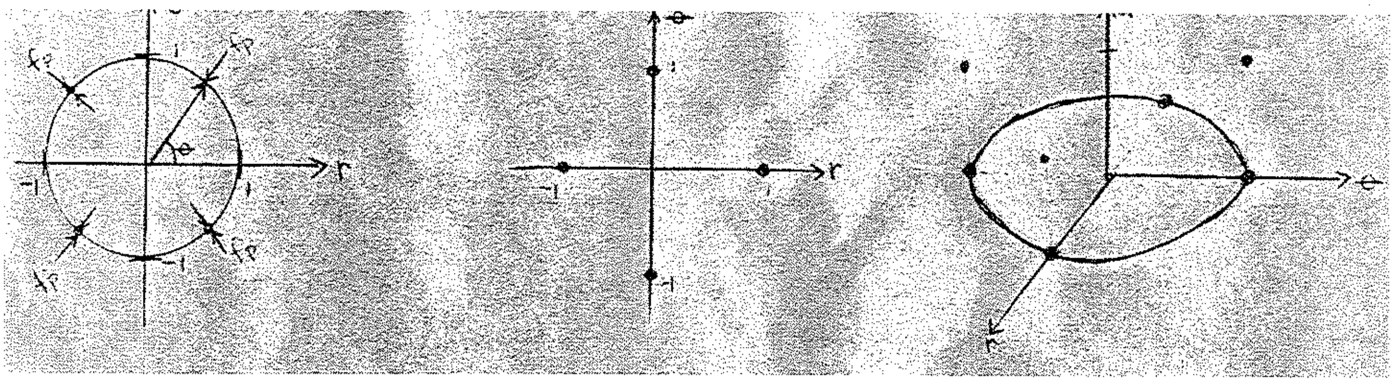


Figure 6: (left) Fixed points for $a = 0$. (middle) Fixed points for $a > 0$. (right) Plot of fixed points with increasing a

We would now like to examine trajectories closer to the heteroclinic point. For $\omega = .45$, the period 1 orbit can be seen in both phase space and configuration space in Figure 11. If we let $a = .03$ we can see in Figures 12 and 13 four islands with four fixed points at the center as expected. We also observe that we still have our KAM curve and the region about the heteroclinic point is highly stochastic. Figure 14 shows the four fixed point trajectories in configuration space. Looking specifcally at one of the fixed points at different values of a produces Figure 15. This is similar to our example, but in this case our fixed points are not constant as a increases but rather are some function of a. As we move to a period 2 orbit, it turns out that, at $a = .004$, we can not only see the islands corresponding to the period 2 orbit but we can also see two period 1 islands corresponding to the pendulum swinging over the top, see Figures 16 and 17.

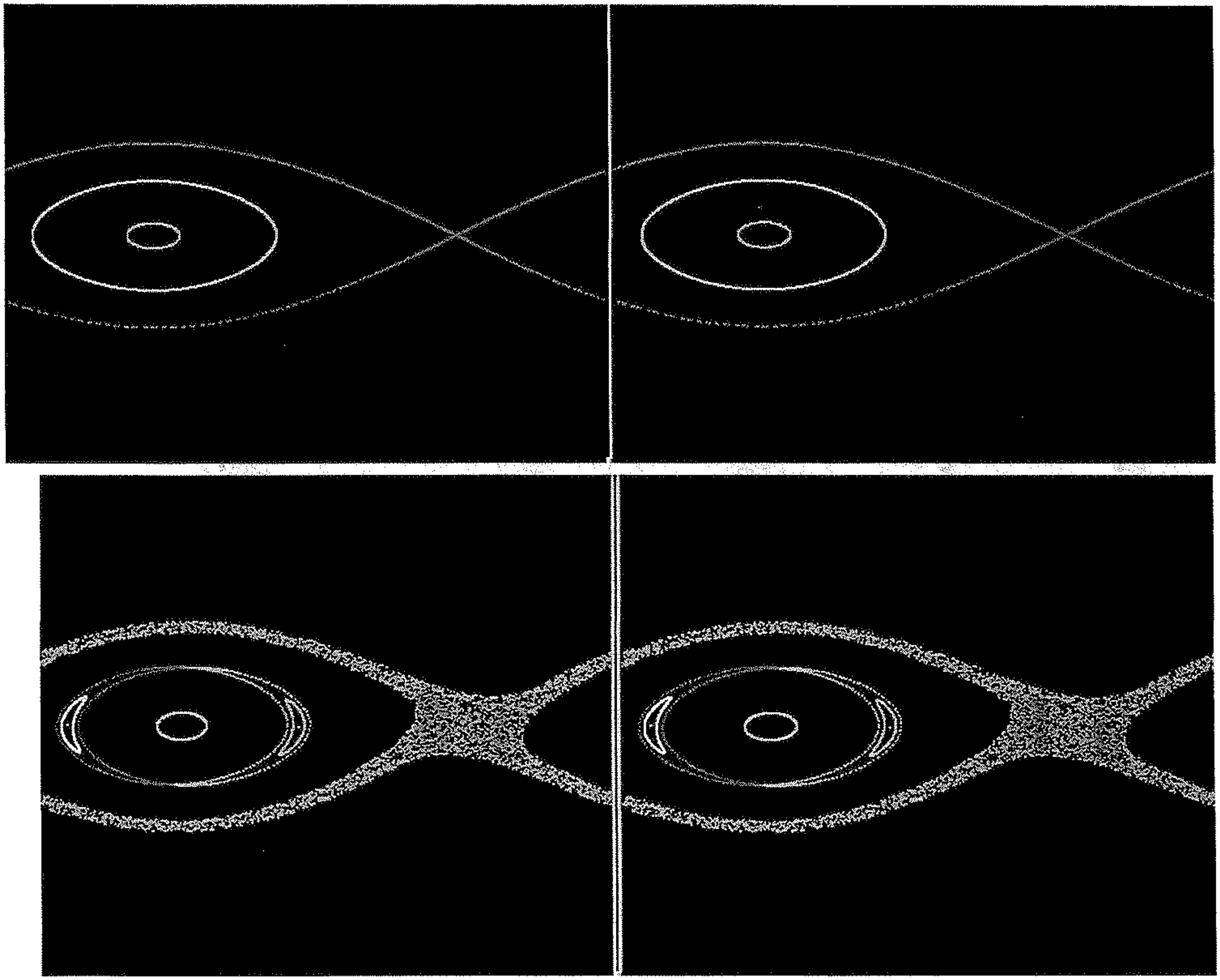
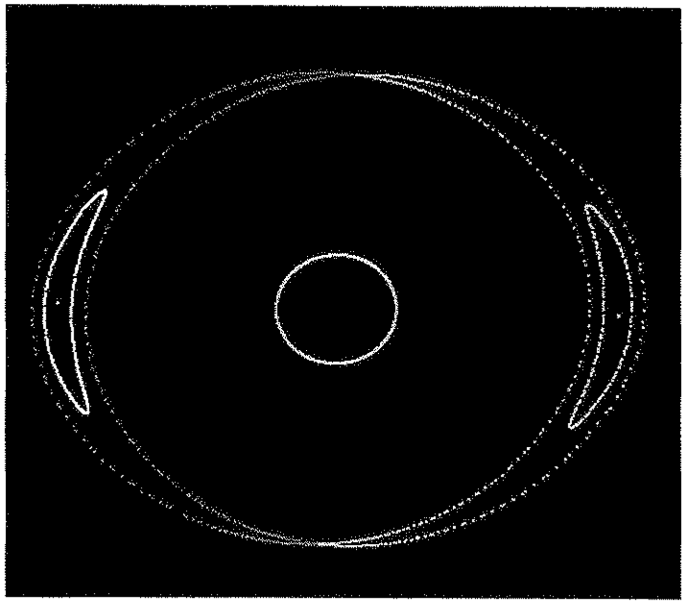


Figure 7: (top) For $a = 0$, The phase space view of a potential KAM curve, the infinite number of period 1 orbits and the trajectory near the heteroclinic point.

Figure 8: (bottom) For $a = .08$, the phase space view of a potential KAM curve, the four remaining period 1 fixed points and the now stochastic trajectory near the heteroclinic point



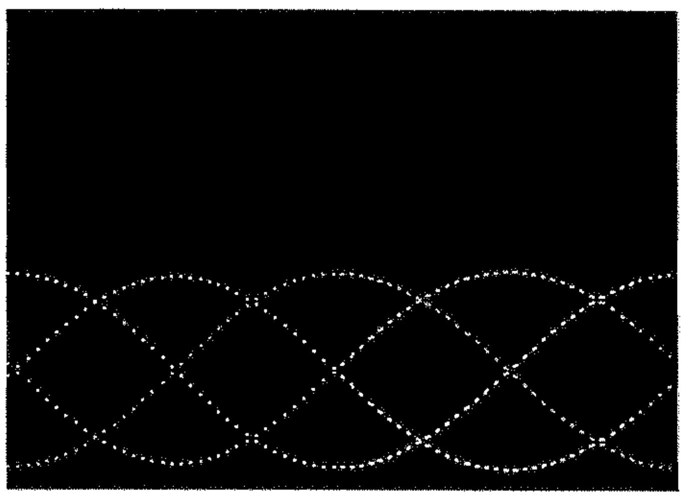
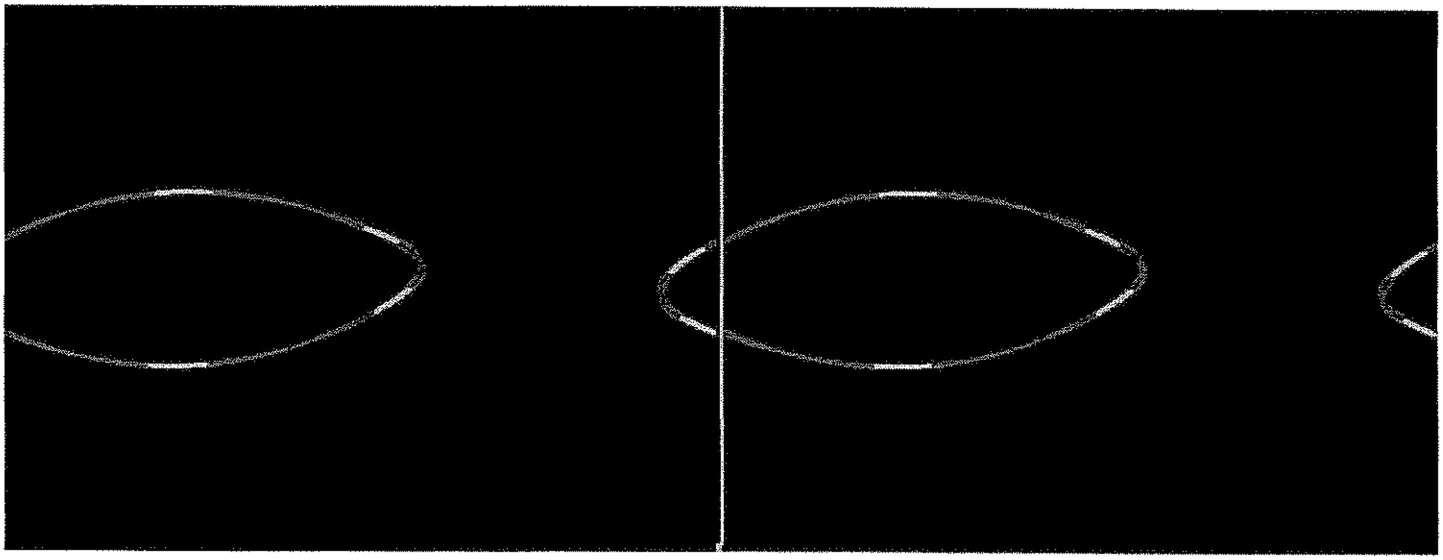


Figure 9: (top) Blow up of islands from Figure 5. (blue) A potential KAM curve, (red and purple) Almost period 1 orbits very near the unstable fixed points. (dots at the centers of the yellow and orange "bannanas") Stable fixed points.

Figure 10: (bottom) Configuration space view of the four remaing fixed point trajectories. The light yellow trajectory corresponds to the left stable fixed point and the blue to the right stable fixed point. The white trajectory corresponds to the bottom unstable fixed point and the green to the top unstable fixed point.



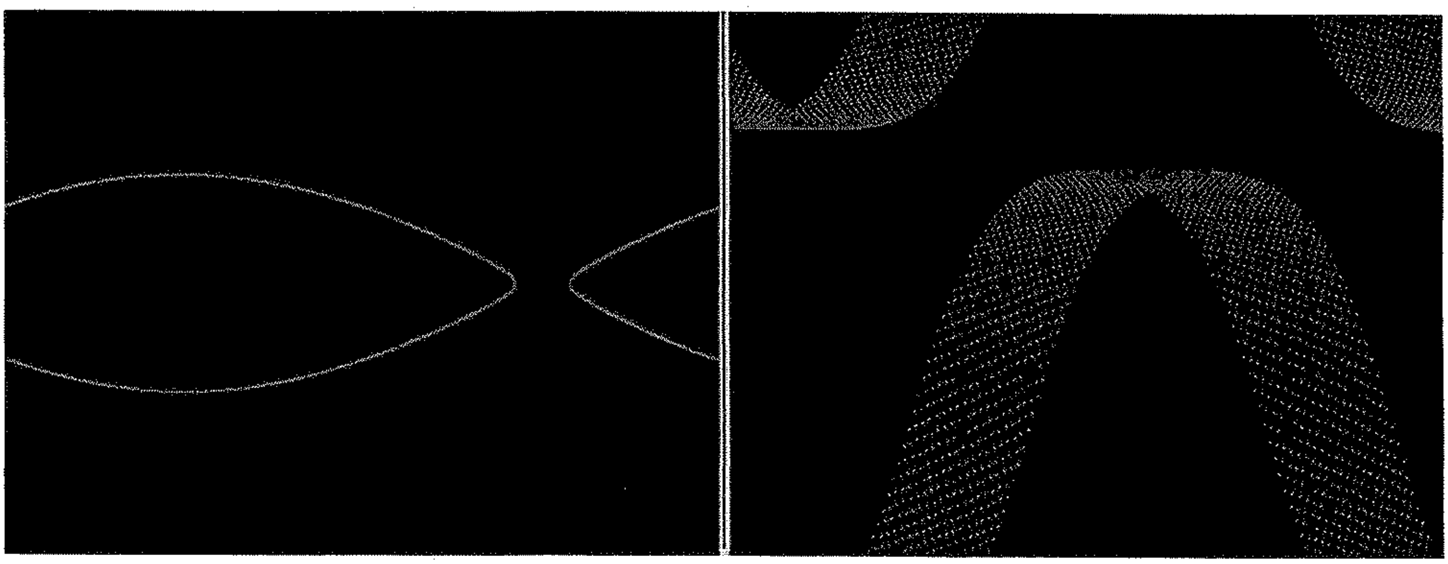
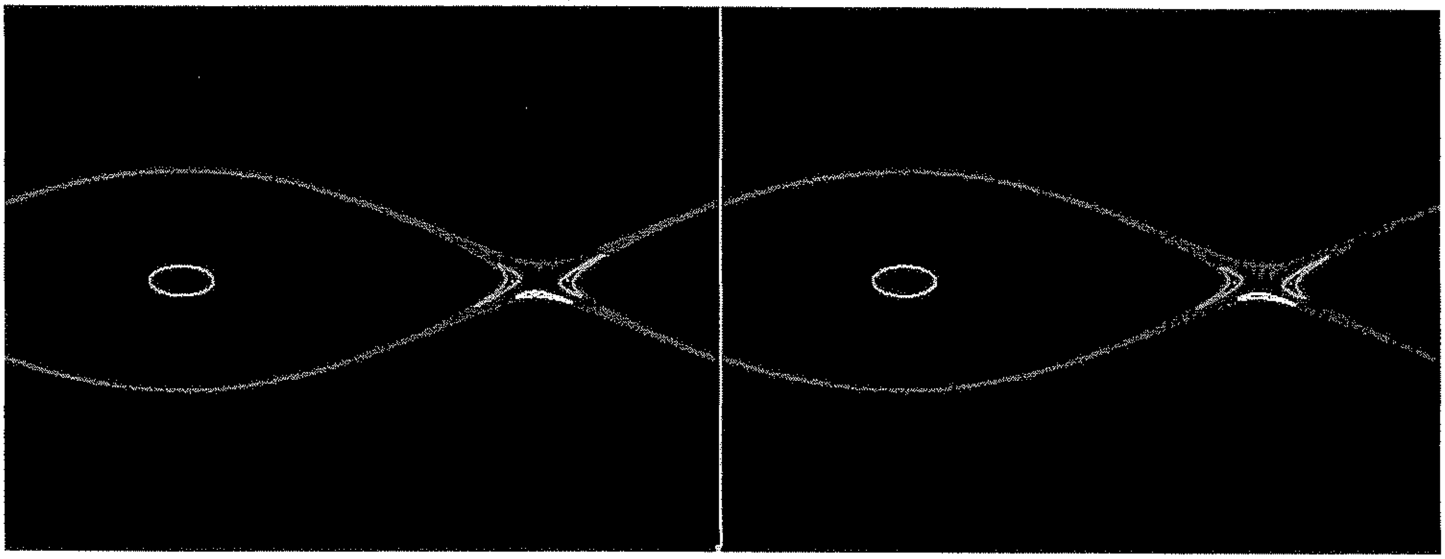


Figure 11: Period 6 island chain, $\frac{w_n}{w} = \frac{5}{6}$, for $a = .24$

Figure 12: (left) The infinite number of period 1 phase space trajectories for $\omega = .45$. (right) Some of the period 1 orbits in configuration space.



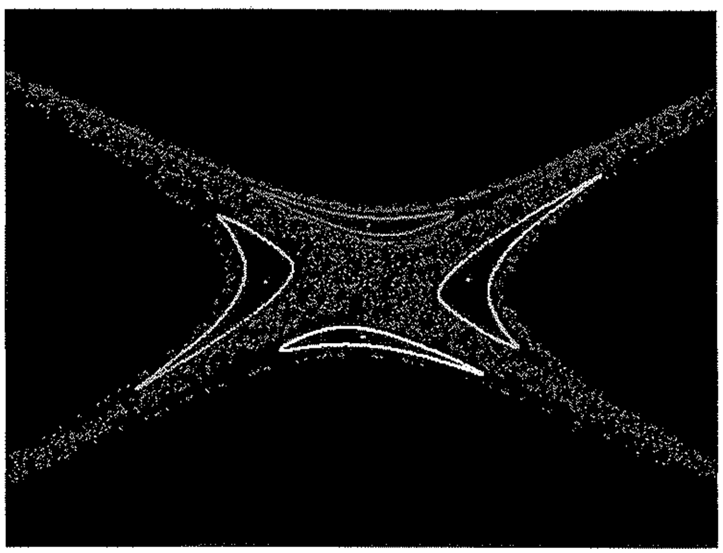
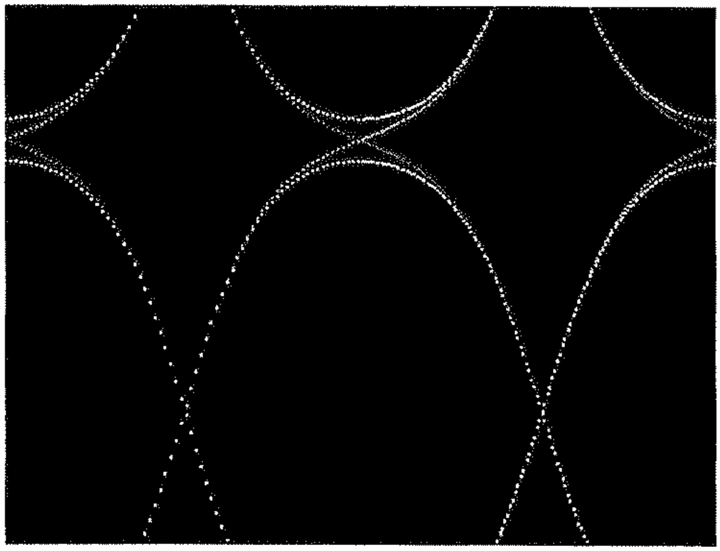


Figure 13: (top) Period 1 islands for $\omega = .45$ and $a = .03$

Figure 14: (bottom) Magnified Period 1 islands



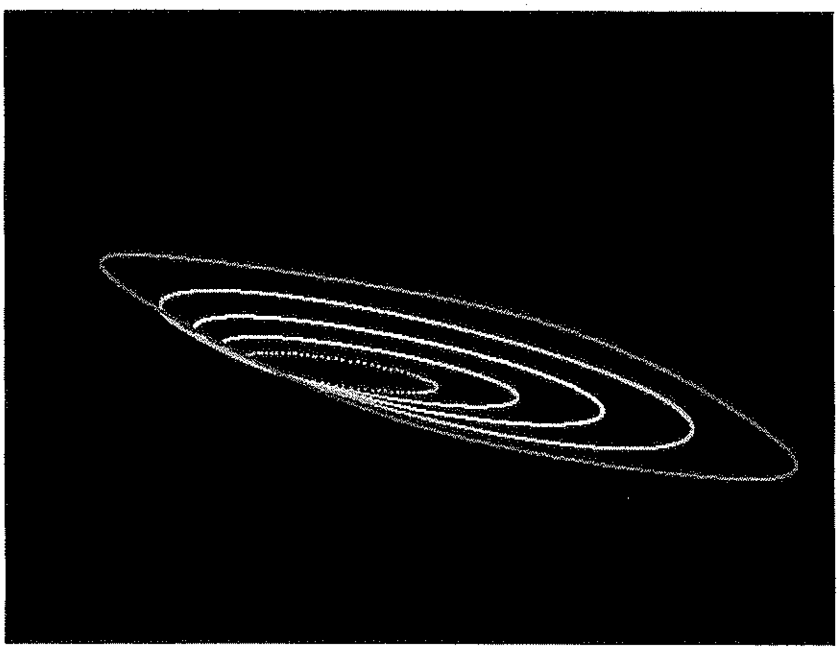


Figure 15: The four fixed point trajectories in configuration space. The yellow orbit corresponds to the right stable fixed point while the light blue is the left stable fixed point. The purple orbit corresponds to the top unstable fixed point while the red orbit is the bottom unstable fixed point.

Figure 16: The fixed point trajectory in phase space for values of $0.04 < a < 0.06$ incremented by .05

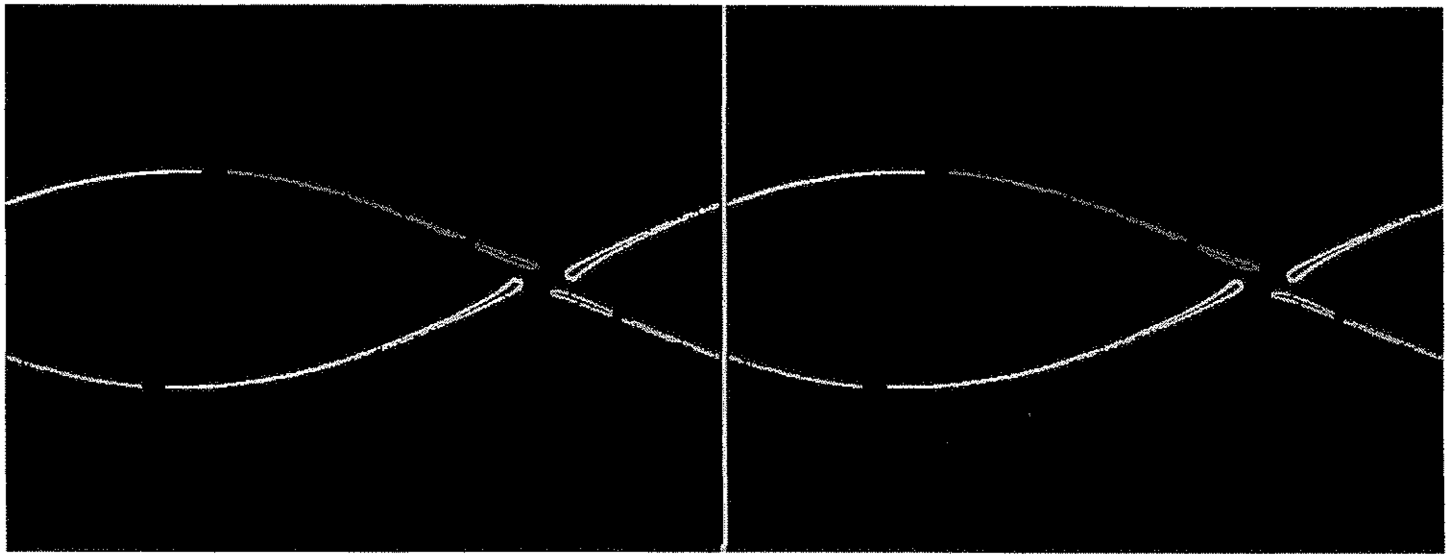


Figure 17: (top) Period 2 islands, $\omega = .9$ and $a = .004$

Figure 18: (bottom) Magnified Period 2 islands

6 The Program and Trajectory Data

The orbits were generated with the program *dstool* using a parsed dynamical system. For the forced pendulum our parsed dynamical system follows directly from eq. (35)

$$\begin{array}{rcl} \dot{\text{theta}} & = & \text{thetadot} \\ \dot{\text{thetadot}} & = & -(1 + a\cos(2\pi\,\text{tau}))\sin(\text{theta}) - \text{nu}\,\dot{\text{theta}} \\ \dot{\tau} & = & 2\pi\,\text{omega} \\ \text{INITIAL} & & \text{theta 3 thetadot 0 tau 0 a 2 omega 1 nu 0} \\ \text{PERIODIC} & & \text{theta=-1.570796327 4.71238898 tau 0 1} \\ \text{RANGE} & & \text{theta=-1.570796327 4.71238898 thetadot -5 5 tau 0 1} \end{array}$$

Period 1, $\omega = 0.9$ and $a = .08$. Figures 4,6,7,8 and 9

	KAM curve	Heteroclinic curve	Stable curve	Unstable curve
θ	-.2701484	3.1402058	1.2701484	.074683519
$\dot{\theta}$	.029480969	.0025501831	-.029480969	1.176186

Period 1[2] , $\omega = 0.45$ and $a = .03$. Figures 11,12,13,14 and 15

	Left Stable Curve	Left Fixed Point	Top Unstable Curve	Top Fixed Point
θ	2.8338433	2.8946973	3.1904417	3.1472661
$\dot{\theta}$	.13161641	-.0065904188	.19998828	.24175529

Period 2, $\omega = 0.9$ and $a = .004$. Figures 16 and 17

	Period 2	Unstable Period 1	Stable Period 1
θ	3.3633559	3.0779501	-3.0779501
$\dot{\theta}$	.010510625	.24128179	-.24128179

[2]To get the corresponding bottom and right curves and fixed points take the negative of the values above.

Period 6, $\omega = 0.9$ and $a = .25$. Figure 10

	Trajectory
θ	2.092463
$\dot{\theta}$	.21920668

7 Conclusions

In this paper we have examined a spring-mass system from both the perspectives of Newtons laws and Lagrangians. The theory of Lagrangians is important in the study of KAM theory in that Hamiltonians follow directly from Lagrangians [3] where Hamiltonian theory is mostly used when one talks about KAM theory. A necessary condition to observe KAM structure is a second order differential equation with a time-dependent driving term. Rather than use the spring-mass system we instead used a forced pendulum. This system nicely showed the relationship between not only island formation and commensurate frequencies but also with KAM curves and incommensurate frequencies. The stochastic nature of the pendulum hanging straight up and down could also be easily seen.

8 Acknowledgments

I would like to thank Dr. Jim Swift his time, patience and guidance in this project. I would particularly like to thank him for pointing out the example used in section 5. Simple as it may be, it cleared up alot.

[3] see Appendix for a proof of Hamiltons equations from the Lagrangian

Appendix

In this appendix we relate the Lagrangian in generalized coordinates, $L(q, \dot{q}, t)$, to a new function $H(q, p, t)$ called the Hamiltonian which describes the sum of the kinetic and potential energies. That is

$$H = T + U = \text{constant}.$$

For a proof see [3, page 265]. In order to relate the Lagrangian with the Hamiltonian it will be necessary to examine the Legendre transformation [1, page 62]. Basically. this transformation is used to express a function $f(x)$ as a new function $g(p)$ where p is a new variable. To construct this new function we define

$$F(p, x) = px - f(x)$$

and define $x = x(p)$ to be the point such that $F(p, x)$ is maximized. It follows that

$$g(p) = F(p, x(p)) = px(p) - f(x(p)).$$

Notice that we have taken a function of x and transformed it into a new function of p. As an example we consider $f(x) = x^2$. From above, it follows that

$$F(p, x) = px - x^2. \tag{36}$$

To find $x = x(p)$ we need to maximize F. This implies

$$\frac{dF}{dx} = p - 2x = 0$$

and thus $x(p) = \frac{1}{2}p$ which we substitute into eq. (32) to yield

$$\begin{aligned} g(p) &= \frac{1}{2}p^2 - \frac{1}{4}p^2 \\ &= \frac{1}{4}p^2. \end{aligned}$$

We now apply the Legendre transformation with respect to $\dot{q}$ to $L(q, \dot{q}, t)$ which yields

$$H(q, p, t) = p\dot{q} - L(q, \dot{q}, t)$$

from which is follows that

$$p = \frac{\partial L}{\partial \dot{q}} \tag{37}$$

which is an expression for the generalized momentum.

If we were to apply the Legendre transformation with respect to q then we could express the Lagrangian $L(q, \dot{q}, t)$ as a function of some new variable $\dot{p}$. That is,

$$H(\dot{p}, \dot{q}, t) = \dot{p}q - L(q, \dot{q}, t)$$

from which it follows that

$$\dot{p} = \frac{\partial L}{\partial \dot{q}} \tag{38}$$

which is the expression for the generalized force. Using eq. (33) we can express $\dot{q} = \dot{q}(p)$, analagous to $x = x(p)$. Taking the total differential of the Hamiltonian yields

$$dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt. \tag{39}$$

Taking the total differential of $p\dot{q} - L(q, \dot{q}, t)$ yields

$$dH = \dot{q} dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt. \tag{40}$$

Equating eq. (34) with eq. (35) and substituting the expression in eq. (34) yields Hamiltons equations of motion

$$\dot{q} = \tfrac{\partial H}{\partial p} \quad \tfrac{\partial H}{\partial q} = -\tfrac{\partial L}{\partial q} = -\dot{p} \quad \tfrac{\partial H}{\partial t} = -\tfrac{\partial L}{\partial t}.$$

References

[1] V.I. Arnold. *Mathematical Methods of Classical Mechanics, Second Edition.* Springer-Verlag, New York, 1989.

[2] A.J. Lichtenberg and M.A. Lieberman. *Regular and Stochastic Motion.* Springer-Verlag, New York, 1983.

[3] Jerry B. Marion and Stephen T. Thornton. *Classical Dynamics.* Saunders College Publishing, Fort Worth, 1995.

[4] Stephen H. Strogatz. *Nonlinear Dynamics and Chaos.* Addison-Wesley Publishing Company, Reading, Massachusetts, 1994.