The Peanut, Potato Chip, and the Infinite Dimensional Volcano

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1 Preface

This paper is the result of a Research Experience for Undergraguates at Northern Arizona University, by John Neuberger Ph.D., and Joel Fish. This paper explores an 80 year old problem in mathematics, specifically finding solutions to the non-linear elliptic partial differential equation: $\Delta u + f(u) = 0$ on a piecewise smooth bounded region Ω, with u=0 on $\partial\Omega$, where Δ is the Laplacian, and f to be a super-linear, sub-critical function.

This paper attempts to shed some light on the problem, and to find new methods of proving existance of solutions.

This paper has also been written specifically for undergraduates. That is to say, the material in this paper is written such that any undergraduate with the appropriate prerequisite material (described below) shall be able to comprehend the mathematics that follows.

2 Preliminaries

We shall assume that the reader has a basic understanding of linear algebra. Such things one should know include but is not limited to abstract vector spaces, linear independance, a basis, eigen values eigen vectors, orthonormal eigen basis, diagnalization, inner products, dimension, isomorphism etc. As well as basic results from calculus: partial derivatives, taylor series etc.

Our first major concept is that of a function space. Consider P_n the collection of polynomials of degree n or less. Clearly P_n is a vector space, with a basis $\{x_i\}_{i=0}^{n}$. We say P_n is a function space since it is a vector space of functions. However P_n is a rather simple function space since it is of finite dimension. We would like to motivate the idea of a function space with infinite dimensions.

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Consider C^∞ the collection of infinitely differentiable functions on the real line. From calculus, as well as complex analysis, we know that for any given function $f \in C^\infty$ there exists a taylor series such that $f = \sum_{i=0}^{\infty} c_i x^i$. Thus we can think of the collection $\{x_i\}$ as an infinite basis for C^∞. We note that we are skipping over some important details here, like whether or not any infinite linear combination of $\{x_i\}$ is in C^∞ (which we know is not true) however we note that we are simply to build a mental picture of an infinite basis of a function space.

The most important result from our above discussion is that once we have chosen our basis, we find and isomophism from an n-dimensional vector space, to R^n. Furthermore, we can find an isomporphism between a function space and $U \subseteq R^\infty$. Once we have found our isomorphism we can think of functions in our function space as points in R^∞. Thus from now on, writing *points* will be synonomous with writing *function*.

3 Functional Intro

The first question we must answer is **What is a functional?** Simply put, a functional is a function of functions. That is to say

Definition 1 *Given a function space X and a map $J : X \to R$ then J is said to be a* functional.

A simple example of a functional might be $J : P_n \to R$ defined to be $J(f(x) = f(2)$. Thus with for every polynomial of degree n or less, we can use J to associate a single real number with each polynomial. Furthermore, once we chosen a basis for P_n, we can use results from our previous section to view J as a map from R^n to the real numbers. This is important, because it is easier to "see" our functional mapping from R^n to R rather than any given function space to R.

Of course such our example is not particularly intersesting. Thus as one can guess, the choice of a functional is entirely dependant upon what problem we are trying to solve. The branch of mathematics concerning itself with chosing functionals, and doing analysis upon them, is called The Calculus of Variations.

Example 1 *Suppose we wish to solve the ordinary differential equation given by*

$$\frac{\partial}{\partial x} f = f$$

on the interval $[0, 1]$ *and did not know that the solution was of the form $f'(x) = Af(x)$ (with A a constant). Then we may choose to study the functional $J : C^1 \to R$ defined by*

$$J(f) = \int_0^1 (\frac{\partial}{\partial x} f(x) - f(x))^2 \, dx$$

Now suppose that somehow we find a function f_0 such that $J(f_0) = 0$. Then since the integrand of J is always greater than or equal to zero, it must be the case that $(\frac{\partial}{\partial x} f_0(x) - f_0(x))^2 = 0$, it quickly follows that

$$\frac{\partial}{\partial x} f_0(x) = f_0(x)$$

Thus we have found a solution to our problem.

Note that in the above example, we have not found *all* solutions, but just rather just one of possibly many solutions.

Although the above functional has some uses, previous studies have shown that it is quite limited. Thus we choose a more interesting functional:

$$J(u) = \int_{\Omega} |\nabla u|^2/2 - F(u)dx$$

where ∇u is the vector gradient of u and $F(s) = \int_0^s f(t)dt$. Solutions to our PDE are no longer given as zeros of J but rather as critical points of J. Note that from calculus we know that a critical point of a function $r : R^2 \to R$ need not be a relative maximum or minimum, but could be a saddle point. In higher dimensions, there are many types of saddle points, and each type will correspond to a different type of solution to our PDE.

We now point out some computational limitations. Suppose we simply wanted to approximate a continuous function defined on the interval $[0, 1]$, with linear interpolation of, say, 100 points. Our intuition should tell us that for most functions, we should have a pretty reasonable approximation for our function. Now suppose our function was defined to map the unit cube to the real line. To achieve the same level of approximation for our new function, we must use a million points. Clearly, for numerical approximations, the greater the dimension, the more time it will take to complete computations. Thus throughout the rest of this paper we will work in the 1-dimensional case. In other words we shall study the ordinary differential equation and boundry value problem given by $y'' + cy + y^3 = 0$ on $[0, 1]$ with $y(0) = y(1) = 0$. Our new functional is then given by

$$J(y) = \int_0^1 (y'(x))^2/2 - y(x)^4/4 - cy(x)^2/2$$

We point out though that our ultimate goal is to make new progress in the 80 year old problem described above. Thus from now on, although we will work in the 1-dimensional case, we shall be extending all our ideas into the n-dimensional case.

4 What Our Functional Looks Like

We are now able to begin our investigations into our problem. Our first task is to determine what the functional "looks like." That is to say, we wish to find out if it has any nice shape, or patterns to it. Fortunately much of this work has already been done, so that we may simply reiterate up those results. We note though, that we are attempting to "see a surface" in infinite dimensions. Thus even though results are known, we must take a moment to visualize our functional.

Our functional can best be described as an infinite dimensional volcano, with center at zero. We shall explore this idea by building up one dimension at a time. We characterize the functional's volcano-ness by the following trait:

Definition 2 *A function (or functional) J is said to be* shaped like an n-dimensional volcano *if $J : V^n \to R$ if for any $u_0 \in X^n - \{0\}$ the real valued function $g(x) := J(xu_0)$ has the following properties for $x \geq 0$:*

(a) $g(0) = 0$

(b) $g(x)$ is twice differentiable

(c) there exists a unique x_0 such that $g'(x_0) = 0$; futhermore, $g''(x_0) < 0$ (i.e. g is concave down at x_0)

(d) $g(x) \to -\infty$ as $x \to \infty$

We shall now give examples of some volcano shaped functions. First we consider the function given by the equation

$$j_1(x) = -(x-3)(x+5/2)x^2$$

Clearly j_1 satisfies all our conditions, since their are only two directions to choose from. We also note that nothing in our definition said that the maximums had to be equal, but simply that they existed.

We shall now consider some examples of 2 dimensional volcanoe shaped functions:

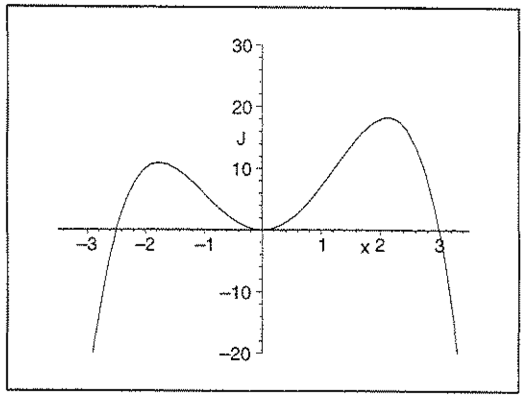


Figure 1: 1D Volcano

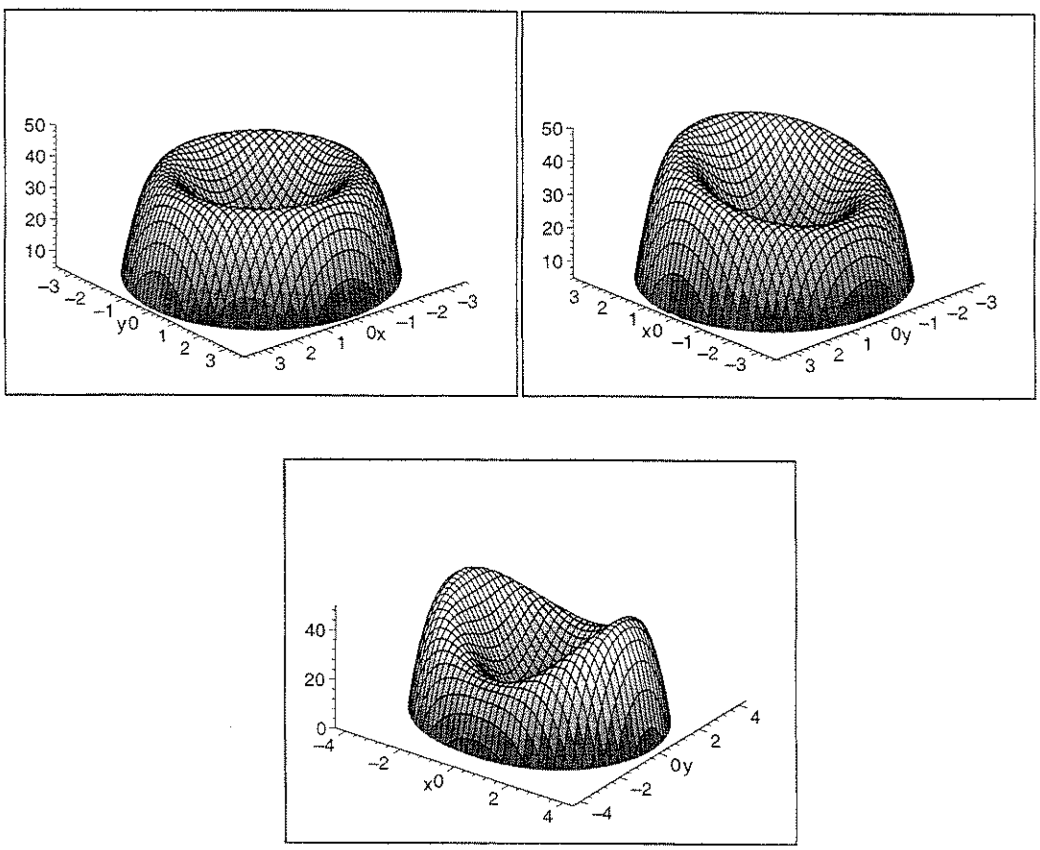
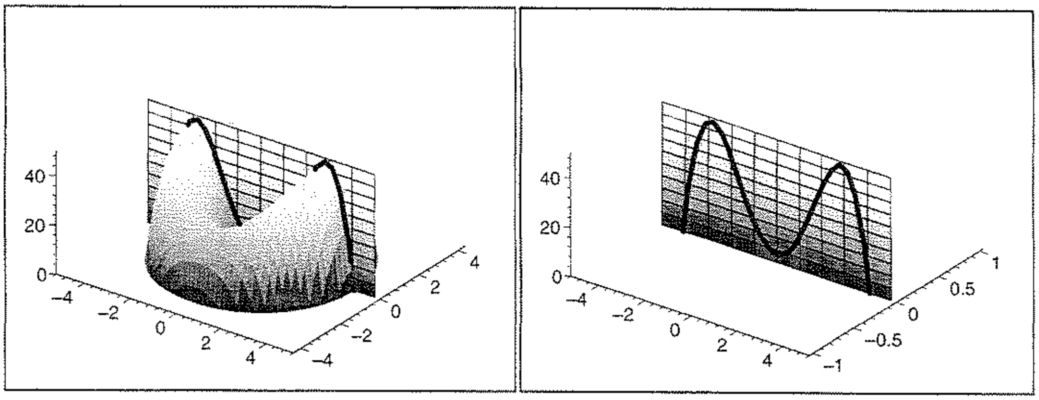


Figure 2: Some 2D Volcanoes: Flat Rim; Slant Rim; Potato Chip Rim

We now discuss the important idea of "taking a slice" of our functional. Consider the 2D potato-chip-rim volcano given by $j_2(x,y) = -(x^2+y^2)^2 + 10(x^2+y^2) + 4y^2$. Now suppose we restrict this function defined on R^2 so that it is defined only on some line in R^2. See below for a graphical representation of this process.



We note that we can generate a 1D volcano from a 2D volcano. This is nice, since a function on R is much simpler to work with than a function on R^2 (or R^n for that matter). Unfortunately, it is clear that we are losing a lot of information about our functional in the process. However, it is possible to take a bigger slice. Suppose we have a function $j_3 : R^3 \to R$. Such a function is very difficult to visualize. Thus we could take a 1-dimensional slice of it in the ψ_1 direction. Furthermore, consider the plane in R^3 given by the span of ψ_1, ψ_2. Certainly j_3 is defined on this plane. Thus if we restrict j_3 to this plane, and plot it, then we have taken a 2 dimensional slice of a 3 dimensional volcano. This is the concept used throughout this paper: J is an infinite dimensional volcano; we approximate it with j_7 (i.e. the functional defined on the span of the first seven basis functions); we make computations, and then display our results by taking 2 dimensional slices through subsets of R^7.

5 Newtons Method

We will now take a short break from working with our functional to review some material. We first consider a standard result from Calculus, Newton's Method. The rough idea of it, is that, suppose we have some differentiable function $g(x)$, and we wish to find x_0 such that $g(x_0) = 0$. Then, if we choose a starting point x_1 to be "close" to a zero of g, then the following algorithm will find a zero of g:

choose x_1

fix $\epsilon > 0$
initialize $i = 1$
Repeat Loop Until $g(x_i) < \epsilon$
Begin Loop
$x_{i+1} = x_i - g(x_i)/g'(x_i)$
increment i
End Loop

It turns out, that in most cases, for an arbitrary choice of x_1, $\{x_i\}$, will converge to a zero of our function. However, there are still several types of problems which can occur in our algorithm. Before we focus on what can go wrong, we would like to point out the many good aspects of Newton's Method. We begin with the description of an ideal case.
Although Newton's Method appears to just be a simple root finding algorithm, it has amazing power when we jump to higher dimensions. Consider needing to find the zero of a a function $g : R^N \to R^N$. Suprisingly, Newton's Method can help us. The following algorithm, again chosing X^1 to be close to a zero, will converge to a solution:

choose X^1
fix $\epsilon > 0$
initialize $i = 1$
Repeat Loop Until $||g(X^i)|| < \epsilon$
Begin Loop
$X^{i+1} = X^i - (g'(X^i))^{-1} g(X^i)$
increment i
End Loop

With $||\cdot||$ as the standard euclidean norm. We should also point out the the "derivative" of g can be represented as an $N \times N$ matrix. The matrix is defined to be

$$\begin{pmatrix} \frac{\partial}{\partial X_1} g(X)_1 & \cdots & \frac{\partial}{\partial X_1} g(X)_N \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial X_N} g(X)_1 & \cdots & \frac{\partial}{\partial X_N} g(X)_N \end{pmatrix}$$

6 Using Newtons Method On Our Functional

We shall now note some important results:

1) There is a well defined first derivative of J.

2) There is a well defined second derivative of J.

The first derivative of J turns out to simply be the gradient of J. We note that this gradient $\nabla J : U \to R$ where U is isomorphic to a subset of R^{∞}. Futhermore, we are looking for critical points of J, which are precisely those points such that $\nabla J = 0$ (note that this is the zero vector and not the zero scalar). Thus in some sense, we are looking for zeros of a function defined to mape $R^{\infty} \to R$ (namely ∇J). We hope to be able to use Newton's Method to find such points, however, we need the existance of the derivative of ∇J (i.e. the second derivative of J). It turns out to be the case that the second derivative of J exists, and is given by the hessian of J. (For those readers who wish to see proofs of the existence of the first and second derivatives of J, we refer you to Dr. Neuberger's CCN paper where these details are given.) Thus, we may now use Newton's Method to find critical points our functional, which are precisely solutions to our ODE/PDE.

We note that there will still be problems using Newton's Method on our functional. For example, if the hessian is not invertible at some point, then Newton's Method won't be defined at said point. Futhermore, it could still be the case that Newton's Method may (1) not converge at all, or (2) converge to a point which is not a solutions. We shall dicuss these problems in the remaining sections.

7 Putting it all Together

We shall now put all our previous discussions together and describe our results. We begin by noting that the Hessian evaluated at each point will be symetric. Results from Linear Algebra state then that each Hessian will be decomposable into the following matrix product: $E\Lambda E^T$ with Λ a matrix of eigenvalues, and E a matrix of an orthonormal eigenbasis of column vectors. We note that the positive eigenvalues of Λ correspond to parts of the functional J which are concave up in the corresponding eigen direction. Since our functional is volcano-shaped, all of the eigenvalues are a positive where $u = 0$. If we take a 2 dimensional slice of our functional, we will find that the slice looks very much like j_2. Furthermore, if we consider the number of negative eigenvalues along a path which moves from $u = 0$ to $u = \psi_1$-solution to $u = \psi_2$-solution, we see that the two of the eigenvalues of the Hessian must change from positive to negative. Since J is twice differentiable, we expect the eigenvalues of the Hessian to vary continuously. We then consider finding all points which only have one negative eigenvalue, only two negative eigenvalues, etc.

It is known that the *signature* of a matrix is the number of negative eigenvalues. Thus we make a signature plot of a 2-dimensional cross-slice of J. See below.

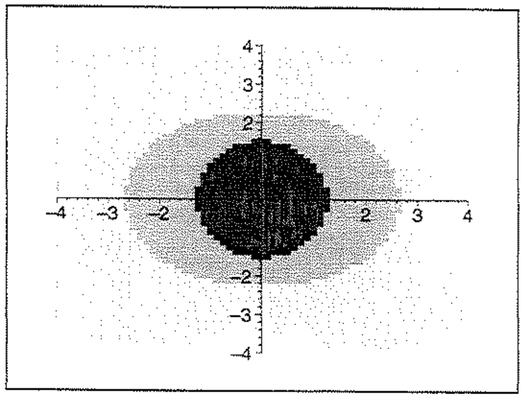


Figure 3: The Signature Plot

We now make a formal definition.

Definition 3 *The inflection set I is defined to be*

$$I = \{p | \lambda_i = 0; i \in \{1, \ldots, N\}\}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix}$$

and $H = E_p^T \Lambda_p E_p$

A more concise definition would be all functions $u \in X$ such that the hessian H at u is singular, however we will need the above definition to discuss some of the properties of the functional J. We see from the signature plot, that part of the inflection set appears to be a collection of sphere-like surfaces, of "dimension" $N - 1$. From the greyscale plot, we see that there don't appear to be any points in the inflection set which aren't seen clearly from the signature plot. Returning to our discussion of Newton's Method, we ask two very important questions:

1) Will either CNM or δNM pass through I

2) Will either CNM or δNM converge to a point in I

Now, it should be clear that if a path generated by CNM passed through an inflection point, the path would pass through a point where CNM was not defined. This is absurd. Similarly, it seems reasonable that if δ is small enough,

δNM should not pass through I either, or at least for most points. From our example in section 7, we see that it is possible for many points to converge to a non-critcal point. We note that our functional has certain restrictions on it which will hopefully prevent our previous example from occuring, however we must give some justification for this hunch.

To test these hypotheses, we generate a plot of the *basins of attraction* with $\delta = 0.5$. We grey-shaded with the following rule: if our δNM starting at point p, converges to a solution with signature zero, p is shaded black; if it converges to a signature one solution, p is shaded dark grey; signature 2 solution, shaded light grey. Here is the basins of attraction next to what we hope it will look like: the signature plot.

We see that the two look similar, but not identical. There happen to even

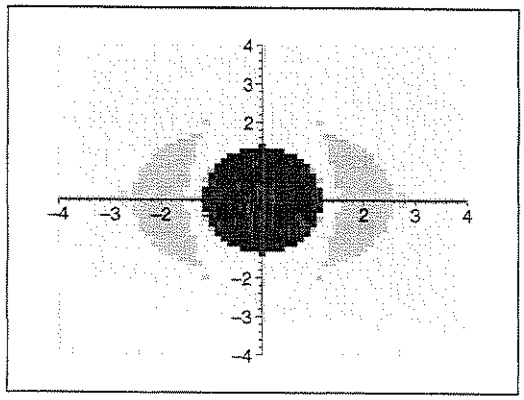


Figure 4: Basins of attraction with $\delta = .5$

be some very strange things happening on the signature 1 and 2 regions. We generate a new plot with a smaller δ in the hope that the strange points are results of a poor approximation to CNM. Fixing $\delta = 0.1$ we generate the following plot:

Now the signature plot and basins of attraction match up identically except the for the two straight lines. [Site Dr. Neuberger Sr.'s work on fractals generated with Newton's Method and how they get smaller for the smaller choices of δ] Thus as we had hoped, Newton's Method appears to successfully converge to a solution for almost every point in the same signature region as it starts.

We now turn our attention to the line of points which don't seem to do what they should. To help us get an understanding for what Newton's Method is doing, we create and plot the vector field defined by $V(p) := -H^{-1}G$ where H and G are the Hessian and gradiant calculated at the point p. Recall Newton's Method and rewrite using V to give $V(p) = p_{i+1} - p_i$. Thus V gives us the

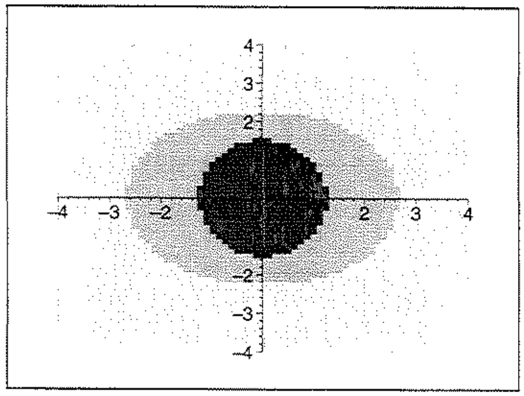


Figure 5: Basins of attraction with $\delta = .1$

direction Newton's Method will will go on the next iteration. We display the plot of $V(p)$ overlayed on our signature plot.

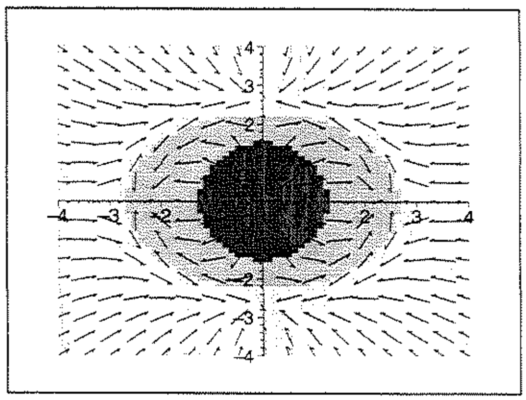


Figure 6: Our vector field overlaid on the signature plot

Note, this is not a true plot of V, but rather its unitized plot: $V(p)/||V(p)||$. This was done since across our region of interest, $||V||$ changed too much to get an accurate picture of what was going on. We now take a moment to discuss V and how it supports our previous hypotheses. Note how the direction of the field changes very drastically near a signature change. This direction change is more reasonable justification for δNM not crossing an inflection set for smaller and smaller δ. I.e. for a big δ at a point p near a signature change could easily be iterated to a point on the other side of the inflection set. Another concern at this point is whether or not V is continuous. We note that the plot is of normalized V, thus as an eigenvalue of H changes from positive to negative (continuously),

V will appear to be discontinuous. We also note that since $J \in C^2$ we expect both H and G to vary continously over X, thus V should also be continuous.

We now discuss the unruley lines which do not do what we expect them to do. First, note that from FIGURE SOMETHING we see that there are two solutions in the same signature region to which Newton's Method converges. Thus there must be some sort of dividing line. Looking at the basins of attraction and the plot of V, we see that our troublesome line happens to be the dividing line between the two basins of attraction. We consider a particular scenario:
Note that for our polynomial "functional" j_2 is symetric across both the x-axis and the y-axis. Thus, consider a point p_0 in the signature 1 region on the y-axis. Now we note that Newton's Method gives us a way to determine which solution to tends towards. However, since we are on the line of symetry, it is not possible to make such a decision since everything is symetric. Since we aren't necessarily at an inflection point, V must still take a value, so it returns a vector which "tends toward both solutions" in the hope that at the next iteration NM will be able to decide which solution is "closer." Unfortunately, at the next iteration p_0 got mapped to p_1 which is still on the y-axis. Again, NM can not decide which solution to tend to, so it heads towards both. Now, repeating this process on j_2 we see that eventually Newton's Method will generate point which get closer and closer to the boundry of the signature regions. We note that δNM will eventually jump across an inflection set. Once it does this, it is allowed to converge to a different solution: one that lies *on* the line of symetry. On the other hand, we have already decided that CNM can not cross an inflection set, so CNM must converge to an inflection point.
Now consider V. We note that for some point p, $V(p)$ is calculated locally. That is, if the function j_2 changed drastically, everywhere except a little ϵ-band around our current line of symetry, then CNM would again take points on the old symetry line to the inflection point, and δNM would take the same points across the inflection set.
As nice as this little theory seems to be, it is a very ideal case. For our functional J, even though a 2d slice of it looks like j_2, it will certainly not be symetric. However, we will still find two solutions in certain sinature regions, and there must still be a dividing "line" of some sort. Perhaps our local symetry idea could still apply. We make the following hypothesis: the dividing line between two basins of attraction in a given by a set of points, all of which are a critical point in an eigen direction. That is, suppose E_i is the set of all eigenvectors of the hessian H evaluated at some point p. Then if p is a point that lies on the dividing line between two basins of attraction of the same signature, then there exists an $i \in N$ such that c where G is the gradient of J evaluated at p.
Before we test this hypothesis, we make a few comments. First we note that this hypothesis clearly encompases our previous symetric case for j_2. Furthermore, since the entries in H are continuous, we expect H to be change in a continuous manor, so then we expect G and the eigenvectors of H to vary continuously. Thus it may be the case that this critical point set is large enough to be interesting. We also note that when we plot this, all our solutions will lie within these

sets, since solutions, by definition, are given as all points such that $< E_i, G >= 0$ for all i. Fixing $\epsilon > 0$ we plot all points such that $| < E_i, G > | < \epsilon$ for some i.

PLOT OF $| < E_i, G > | < \epsilon$

8 Conjectures

8.1 The set of hyper surfaces Q

We define

$$Q = \{p | E_i(p) G(p)^T = 0\}$$

The eigen vectors of each hessian, at each point, can be ordered. Furthremore we can order them such that given $E_i(p)$ as the ith eigen vector at point p, $E_i(p)$ will vary continously as p varies over U.
The remaining assumptions assume such an ordering has been given.

(Q1) Q is connected.

(Q2) Q can be expressed as $N + 1$ non-disjoint subsets $S_0 ... S_N$

Furthermore, $S_0 ... S_N$ have the following properties:

(S1) $S_1 ... S_N$ are plane-like. i.e. an unbounded "sheet-like surface".

(S2) S_0 is sphere-like however it is unbounded

(S3) each set $S_0 ... S_N$ is codimension 1

(S4) each set $S_0 ... S_N$ is a manifold

(S5) no S_i is allowed to self interesect itself (note, this is a property of a manifold)

(S6) given point p in S_i, and eigen vector of the hessian E_i, p is a critical point in the E_i direction

(S7) at any point such that S_i intersects S_j, the intersection must be perpindicular

(S8) a solution exists at any point such that N of the S_i's intersect.

(S9) there exist $2N + 1$ Solutions in the finite case

(S10) CNM is guaranteed to converge to a solution if started at a point $p \notin Q$

We also predict that most of these extend to the infinite dimensional case. We restate a few of the above statements such that that they do not strictly depend on the dimension N.

(S8a) a solution exists at any point p such that $p \in S_i$ for all i except one. That is to say, p is a solutin if and only if there exists exactly one i such that $p \in S_1 \cap S_2 \cap \cdots \cap S_{i-1} \cap S_{i+1} \cap \cdots$
. Furthermore,
(S9a) there exist an infinite number of solutions in the general case

8.2 The set of paths P

We note that it may be the case that even at best, working with infinitely many codimension 1 manifolds, may prove to be a rather daunting task. Thus we consider the next most logical set:

$$P = \{p | p \in S_1 \cap S_2 \cap \cdots \cap S_{i-1} \cap S_{i+1} \cap \cdots S_{j-1} \cap S_{j+1} \cap \cdots\}$$

with $0 \leq i < j$.
We note that P itself in not as interesting. However we believe P can be broken up into a rather interesting collection of sets. More formally we predict that P can be represented as a countably infinite set of paths $\{P_i\}$ (important note each P_i has dimension 1). Furthermore, $\{P_i\}$ has the following interesting properties:

(1) any two intersecting paths P_i and P_j, do so either perpindicularly, or tangetially.

(2) at any point such that two paths intersect perpindicularly, a solution exists.

(3) at any solution, N paths extend from the solution

(4) following any path will lead to either another solution, or to infinity.

(5) at any point p on a path that is not a solution, p will be a critical point in all directions v, such that $v\cdot$ tangent(path)$= 0$

8.3 The inflection set I

Definition 4 *The inflection set I is defined to be*

$$I = \{p|\lambda_i = 0; i \in \{1, \dots, N\}\}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix}$$

and $H = E_p^T \Lambda_p E_p$

In other words, I is the set of all points p such that the hessian at p has at least one eigen value of zero.

(I1) The set I, can be represented as a set of sets, such that each set I_i has it's ith eigenvalue as a zero

(I2) Ffor any point p in I_i, E_i is normal to I_i if and only if there exists a path P_i which intersects I_i at p. Note the importance of the ordering on the E_i's

8.4 The ordering of the E_i's

Although I believe all of the above statements to be true regardless of ordering of E_i's, I believe that there is a natural way to order them. The particular case that I looked at for this problem, the eigen directions did not vary too much between any two given points in U. Furthermore, the most natural informal way to order the E_i's was to say that the E_1 would be the eigen vector "closest" to e1, which turned out to be $sin(\pi x)$. E_2, closest e.v. to e2, which was, $sin(2\pi x)$, Etc. This ordering is computationally simple, and it is also based on the fact that there is a solution close to $sin(\pi x)$, and another solution near $sin(2\pi x)$, etc, and that for each solution we find, the greater the n in $sin(n\pi x)$, the "further" the solution is from the origin. Furthermore, $sin(\pi x)$ can be roughly be considered to be a simpler function than $sin(2\pi x)$, so we order our eigenvectors by saying which $sin(n\pi x)$ are you closest too" and then ordering our answers from simplest $sin(n\pi x)$ to most complicated.
The point of this clarification, is that as soon as one jumps from the simple ode case to the PDE case, we have more than one spatial direction to choose from, and we begin coparing our eigen vector functions, to functions like $sin(\pi x)$ and $sin(\pi y)$. Who is to say which is simpler? And if we can't decide which is simpler, then we would have to arbitrarily choose an ordering. From results of my ODE case, I believe that there may be a natural way to order our eigen vectors.

For the ODE case the I_i's appear to be disjoint. If this were the case, we could order the I_i's as shells such that the interior of I_i contains i-1 other inflection sets. Then the E_j's could then be ordered in the following manor:
(1) Choose an inflection set I_i
(2) Pick a point p on I_i
(3) Define the corresponding E_i such that at p, the eigen value associated with E_i is 0.

This will then give a natural ordering to our eigen vectors.
This ordering has several possible weaknesses. First is the relative difficulty in computationally ordering in this manor. Especially since all we really need to do is assign an arbitrary order, use it consistantly, and all the above ideas should still hold. Second, and more importantly, is to consider the case where the I_i's are not disjoint. Such cases appear to be quite possible and even likely in the PDE case. From numerical results I've seen, my above odering can be extended to include possibilities like those in the inflections sets in the diagrams below. Such an extension would be possible, since the I_i's could still be ordered by not requiring proper containment of $i-1$ I_i's inside it. Of course, there is also the posibility that something even more horendous occurs, as in the diagram below. In such a case my ordering may completely fail, or may simply fail to order particular consecutive shells. In any case, further research should be done.

9 Further Topics of Investigation

(1) "animated" plot as λ increases, of the Inflection Set and the Critical Point Set with $f(u) = u^3 + \lambda u$. NOTE that results from this plot could change the ordering of the inflection sets.

(2) Consider restricting the class of functions down to polynomials, and possibly heavisided polynomials, and see if results are provable, specifically starting with the $f(u) = u^3$ case.

(3) Natural orderings of the E_i's using an ordering of the I_i's.

(4) Consider the difference between using paths in P and geodesics.

(5) Consider an $f(u)$ such that $f(u)$ is constant for some small region.

10 Miscellaneous Notes

A quicker way to generate the basins of attraction plots: instead of iterating δNM for each point in the plot region, one could calculate V at each pixel to be colored, and then choose a p_0, use the precalculated $V(p_0)$, and instead of recalculating V at $p_0 + V(p_0)$ we could simply use the value of V at the nearest neighbor (or even truncated neighbor) of $p_0 + V(p_0)$, which has already been calculated.

11 Discussions Needed for Completeness

A brief history of the problem

A glossary

A bibliography

Diagrams showing how Newton's Method progresses in 1 and 2 dimensions

Justifications of all our conjectures

Dr Neuberger's Idea for proving infinitely many solutions using Continuous Newton's method

Detailed explanations of what can go wrong with Newton's Method