Semi-Transitive Orientations of Circulant Graphs

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Abstract

In the following paper we will discuss circulant graphs and the semi-transitive properties of these graphs. We will classify the graphs into families of circulant graphs, and prove some general results about when these families are semi-transitive.

1 Introduction

The subject of semi-transitive circulant graphs is not as well studied as other types of semi-transitive graphs. A circulant graph is a graph represented as $C_N(a_1, a_2, ...a_k)$ where N is the number of vertices and $a_1, ...a_k$ are called jumps of the graph. Edges are defined in the manner, v,w is an edges if $w \equiv v \pm a_i$ mod N for some a_i. To standardize notation, all the jumps a_i of a undirected graph will be integers such that $a_i \leq \frac{N}{2}$, since in the undirected case a_i and $N - a_i$ are equivalent modulo N. A directed circulant graph denoted by $C_N[a_1, ..a_k]$ is defined in the same manner as the undirected above, with the exception that the edges between v and $v + a_i$ mod N is now a directed edge in the direction $v \to v + a_i$. In the directed case jumps will be allowed to be greater than $\frac{N}{2}$, since a_i and $N - a_i$ still form an edge between the same vertices, but the edges are pointed in opposite directions and are thus different edges. An example of a circulant graph and a directed circulant graph ~~is given below~~.

Some important terminology and properties of graphs to our research are automorphisms of a graph, vertex-transitive graphs, edge-transitive graphs, dart-transitive graphs, and semi-transitive graphs.

An **automorphism** of a graph is a permutation of the vertices which maintains the edge structure of a graph. So if σ is the permutation and $\{v, w\}$ is an edge of the graph Γ then $\{v\sigma, w\sigma\}$ is an edge of $\Gamma\sigma$. Like the automorphisms of a group or other structures, the automorphisms of a graph form a group under composition, we will be looking at the automorphism group of a graph denoted by $Aut(\Gamma)$ which is the set of all the automorphisms of a graph.

A **vertex-transitive** graph is a graph where all the vertices can be considered to be in the same equivalence class. An equivalence class can be thought of as an orbit. So if there exists $\sigma \in Aut(\Gamma) \ni v\sigma = w$ then w is in the orbit of v. Then if we consider $Aut(\Gamma)$ acting on the set V(Γ), the set of vertices, then a graph is vertex-transitive (from here on denoted by V-T) when there is only one orbit of the action on the vertices.

Lemma 1.1 *All circulant graphs are vertex-transitive.*

> **Proof:** Let σ be the permutation (1 2 ... N). So this is an automorphism of Γ and by applying σ N-1 times then we can take vertex 1 to every vertex and so Γ is V-T.

Similarly **edge-transitive** graphs are graphs where all edges are in one orbit or equivalence class. If we consider the action of $Aut(\Gamma)$ on the set of edges E(Γ) to be defined as $\{v, w\}\sigma = \{v\sigma, w\sigma\}$ then a graph is edge-transitive (denoted

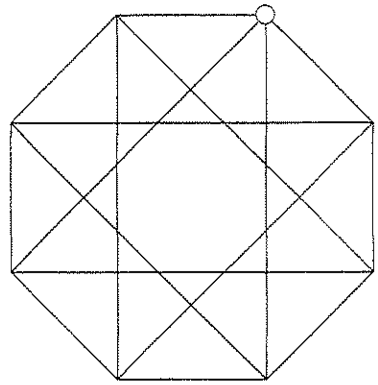

Figure 1: $C_8(1,3)$

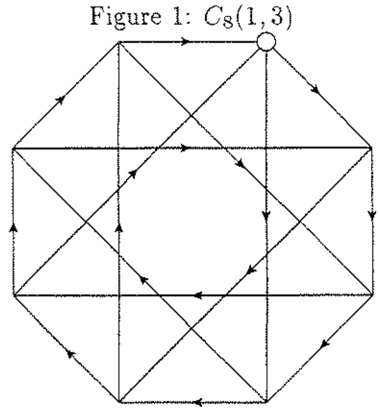

Figure 2: $C_8[1,3]$

E-T) when for all edges $\{a,b\}$ there exists $\sigma \in Aut(\Gamma) \ni \{v,w\}\sigma = \{a,b\}$. So as you can see this definition means that under the action of $Aut(\Gamma)$ there is one orbit in E(Γ).

A **dart** is a half edge or directed edge of a graph. In an undirected graph, each edge is made up of two darts pointed in opposite directions. So if E(Γ) contains the edge $\{v,w\}$ then the set of darts, which we will call D(Γ), contains the darts (v,w) and (w,v). The notation for darts is (v,w), an ordered pair instead of the set $\{v,w\}$ since there is a direction. So if we define the action of $Aut(\Gamma)$ on D(Γ) as we defined it on E(Γ) then a **dart-transitive** (D-T) graph is a graph where there is only one orbit in D(Γ) under this action. The undirected graph shown above is V-T, E-T, and D-T.

A simpler explanation of V-T, E-T, and D-T, is concept of similar. We shall say that vertices v and w are similar if there exists a $\sigma \in Aut(\Gamma) \ni v\sigma = w$, with the same holding for edges and darts. So we will say that a graph is V-T (E-T and D-T in the same manner), if every vertex is similar to every other vertex. When things are similar then they will have the same properties. So if one vertex exhibits a property, and we know the graph is V-T, then we know that every vertex has this property.

The **degree** of a vertex is the number of edges at a vertex. In a directed graph the in-degree is the number of edges coming into a vertex and the out-degree is the number of edges going out of a vertex.

Definition 1.1 *An orientation of a graph is a way of assigning directions to the undirected edges. A regular orientation is a orientation where the in-degree and the out-degree are equal at every vertex and the in-degree of every vertex is equal.*

For the remainder of the paper we will be discussing orientations of circulant graphs. In all cases the orientations will be regular orientations. Also important to our topic is the idea of circulant and noncirculant orientations. A circulant orientation of a circulant graph is an orientation in which every edge of a jump is directed either clockwise or counterclockwise. Another way of defining a circulant orientation is that all the jumps point positively (clockwise) or negatively (counterclockwise). A noncirculant orientation is just the opposite of circulant meaning it has at least one jump in which some of the edges of the jump are directed positively and some are directed negatively. We shall be looking at some circulant and noncirculant orientations of graphs. We have the following lemma concerning circulant orientations.

Lemma 1.2 *If $Aut(\omega)$ contains an n-cycle where n is the number of vertices in the graph then ω is a circulant orientaion.*

> **Proof:** Suppose $Aut(\omega)$ has an n-cycle. Let us relabel the vertices in such a way that the n-cycle is the cycle (1 2 . . . n), a rotation of the n-gon. Further, if a jump points from a to b then by this rotation all jumps must point in the same direction. Thus, ω is circulant by the definition of circulant orientations.

Now let us define the most important concept to our research, semi-transitivity.

Definition 1.2 *A graph Γ is semi-transitive (S-T) if there is a semi-transitive orientation of the graph call it ω which is V-T and E-T. Alternatively a graph is S-T if there exists a subgroup of $Aut(\Gamma)$ such that the subgroup is V-T and E-T on Γ but not D-T.*

From this definition one can see that semi-transitive orientations imply regular orientations. Since a S-T orientation must be V-T, meaning all vertices are alike, one see that the the in-degree and out-degree of a vertex must be equal to the in-degree and out-degree of every other vertex, respectively. Further, if the in-degree of a vertex did not equal the out-degree, then there would be an unequal number of out-edges and in-edges, which can never happen in any directed graph. So every S-T orientation must be regular and thus we will only be looking at such orientations.

2 Known S-T Circulant Graphs

When we began exploring circulant graphs and their semi-transitive properties there were three families of circulant graphs we knew to be semi-transitive. The first family we knew about are the circulant graphs where the jumps form a subgroup of U_N, the multiplicative group where $a \in U_N \Leftrightarrow a < N$ and $gcd(a, N) = 1$.

Theorem 2.1 *If the jumps of ω form a supgroup of U_N, call it S, and $-1 \notin S$ then ω is a S-T orientation of the underlying graph Γ and Γ is a S-T graph.*

> **Proof:** Let ω be the orientation $C_N[a_1, ...a_k]$. Suppose the jumps form the subgroup $S \leq U_N$. Suppose also that $a, -a \in S$. So there is an edge $v \rightarrow v + a$ and an edge $v + a \rightarrow v$, and then there would be an undirected edge between v and $v + a$ and so this in no longer an orientation. Thus, if $a \in S$ then $-a \notin S$ must be true. Since S is a subgroup, $1 \in S$ and so $-1 \notin S$. Now since any element $a \in S$ is relatively prime to N so the jump a will form an n-cycle. So it is apparent that by simple rotations we can send an a edge to any other a edge. Any σ arrange the vertices such that the a edges are sent to the 1 edges on the outside. Let b be another jump of the subgroup. Since a and b are elements of the subgroup, then ba is an element of the subgroup and is thus a jump of the graph. After applying σ we must be sure that there are still the b edges. If we look at vertex v and count down to the b vertex, since the outside vertices move in a steps, so the b vertex under σ is the original ba vertex and there is a ba edge there, meaning σ preserves edges, and so $\sigma \in Aut(\omega)$. We can send any a edge to any other a edge and we can send any x jump into the 1 jumps, thus we can send any edge to any other edge and so ω is E-T. Therefore ω is a S-T orientation of the underlying graph and the graph is S-T.

An example of a graph and its subgroup orientation is given ~~in the following two figures~~ figures 3 and 4.

The next family of S-T circulants we knew about before starting our research was wreath graphs. Wreath graphs are denoted by $W(n, k)$ and are comprised of **n bunches** of **k vertices**. The bunches are arranged around a circle and each vertex of each bunch is connected to every vertex in the two bunches immediately clockwise and counterclockwise to it. A wreath graph is isomorphic to the circulant graph $C_{nk}(b_1, ...b_j)$ where all $b_i \equiv \pm 1$ mod n [Onk95]. ~~Below is~~ ~~an~~ A example of a wreath graph and its circulant equivalent. are shown in figures 5 and 6,

~~**Lemma**~~ Thm **2.1** *Wreath graphs $W(n, k)$ with $n > 2$ are S-T graphs.*

> **Proof:** Suppose we have the wreath graph $\Gamma = W(n, k)$ with $n > 2$. Let ω be an orientation of Γ, and let ω be such that each vertex of a bunch has its edges pointing out to the clockwise bunch and in

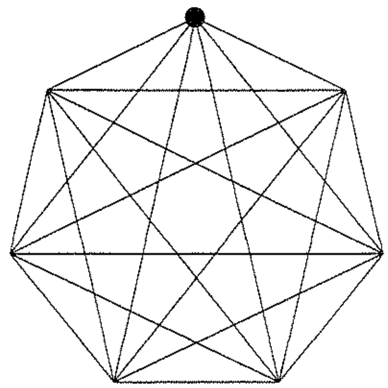

Figure 3: $C_7(1,2,3)$

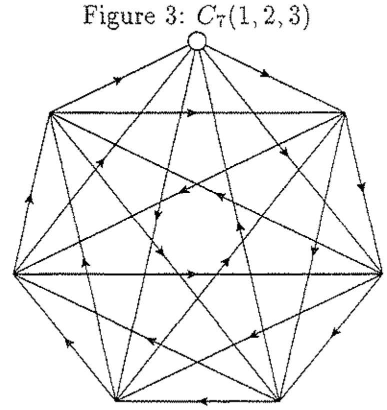

Figure 4: $C_7[1,2,4]$

from the counterclockwise bunch. Each vertex of a bunch has edges pointing to every vertex of the clockwise bunch, then a k-cycle of the vertices in the head bunch will be an automorphism of ω and will be transitive on the edges between the two bunches. Further, the orientation is similar to a clockwise orientation of the n-gon, then through a simple rotation of the bunches we can send edges between bunches 1 and 2 to edges between bunches x and w. Thus we can send any edge to any other edge. Therefore ω is a S-T orientation and $\Gamma = W(n,k)$ is a S-T graph.

The final family of circulant graphs we knew to be S-T are known as the Depleated Wreath Graphs, $DW(n,k)$. A depleated wreath graph is similar to the wreath graph in that it has **n bunches** of **k vertices**, but the edges are defined unlike the edges of the wreath graph. If we label the bunches $0,...n$ and label the vertices in bunch 0 $(01),(02),...(0k)$, and similary for all other bunches, then there exists an edge $(ij)--(i+1k)$ if and only if $j \neq k$. As circulant graphs the graph is denoted as $C_{nk}(b_1, ..., b_m)$ where each $b_i \equiv 1$ mod n.

Lemma 2.2 *All DW(n,k) for $n > 2$ are S-T.*

> **Proof:** Let the orientation of $DW(n,k)$ be a circulant orientation in a clockwise direction. We will first show that the orientation is V-T. Let (i j) denote the vertex in the i^{th} bunch in the j^{th} position. Let ρ be such that (i j)ρ = (i+1 j) and let σ be such that (i j)σ = (i jσ) where σ is a permuation of 1, ..., k. We know that (i j), (i+1 k) is an edge iff j $\neq$ k. We can see that these permutations preserve edges and are thus automorphisms. Since if we apply ρ n-1, times then we will be able to move (i j) to every other (x j) for some x such that $0 \leq$ x $\leq$ n-1 and by applying σ k-1 times we can send every (i j) to every other (i y) where $1 \leq$ y $\leq$ k. So therefore this orientation is V-T. So if we look at the edge {(0 1), (1 2)} and {(0 a), (1 b)} then we define σ by 1σ = a and 2σ = b, so then {(0 1), (1 2)}σ ={(0 a), (1 b)}. So we can permute edges in the bunches and by ρ we can rotate the edges around from bunch to bunch. Therefore the orientation is E-T and $DW(n,k)$ is S-T.

3 Initial Research

Armed with these three known families of S-T circulant graphs, we set out to classify all circulant graphs with $5 \leq N \leq 20$. We first made list of all circulant graphs with vertices between 5 and 20. We eliminated many possibilities of circulant graphs by using the Adam's Isomorphism Theorem which says the following [ET70, 297-307].

Theorem 3.1 *If $gcd(n,m) = 1$ then $C_n(a_1, a_2, ...a_k) \cong C_n(ma_1, ma_2, ...ma_k)$*

A proof of this fact is given in Ricahrd Onkey's 1995 REU paper. This theorem allowed us to eliminate many possibilities of different circulant graphs, by finding all the circulant graphs isomorphic to each other. After creating a list of all possible circulants, excluding isomorphisms, we had 499 graphs to consider. We then set about to find all E-T graphs in this list. Using the program *Groups and Graphs* we determined whether or not the graphs were E-T. First we noticed this following fact about circulants graphs. In any graph if the graph is V-T and the stabalizer group or subgroup of a vertex is transitive on its neighbors (meaning vertices sharing an edge), then the graph is E-T. In the circulant graphs, since we can reverse any edge then it is also true that if the graph is E-T, then the stabalizer of a vertex will act transitively on the neighbors. So using this fact, we looked at the graph, found the automorphism group and then the stabalizer of a vertex and determined if the graph was E-T.

After finding all E-T circulants with $N \leq 20$ we began to look at which of these we knew were S-T because they either had a subgroup orientation or were wreath graphs. We also eliminated E-T circulants which turned out to be disconnected graphs, such as the graph $C_{12}(2,4)$. We eliminated these possibilities since they are multiple copies of smaller graphs, and any thing we would find for the smaller graphs would apply to the bigger graph. At this point we were looking at about 49 possible S-T circulant graphs. After finding all the subgroups and wreath graphs we took this number down to 22 possible S-T circulants and 27 S-T circulant graphs for which we had an orientation and an explanation.

4 Methods of Finding Semi-Transitive Orientations

As we began studying and researching whether or not graphs were S-T, we began to develop and a variety of methods to help us in our work. The most straight forward but least effective method was the plug and chug method. We would try all possible circulant orientations of a graph and see it they were E-T and thus were a S-T orientation of the graph. This method, although tedious and not the most advanced way to think of the orientations, yielded a surprising number of results. Through this method we were able to eliminate possibilities of circulant orientations for a number of graphs.

The next method we brought in to study the graphs is what we call triangle analysis. Triangle analysis, as its name implies, involves studying the triangles of a graph, if there are any. The triangles we look at are either inconsistent or consistent triangles. A consistent triangle is a triangle with a cyclic or circulant orientation. So if vertices a, b, c are in a triangle, then the triangle is consistent if the edges look something like $a \to b$, $b \to c$, and $c \to a$. An inconsistent triangle would then be a triangle would then be a triangle which does not look like this. Moreover an inconsistent triangle is one with edges looking something like this, $a \to b$, $a \to c$, and $b \to c$. An example of a consistent and inconsistent triangle are given in Figure 7.

The way to study these triangles is to count them and then use the following fact.

Lemma 4.1 *If an orientation ω is S-T, then if E is the number of edges and T is the number of inconsistent or consistent triangles, it must be true that $E \mid 3 * T$.*

> **Proof:** Let ω be S-T and suppose E does not divide $3 * T$. Since every edge is alike, each must be a part of the same number of inconsistent and consistent triangles. Let T be the number of consistent triangles. If E is the number of edges, then T_E, the number of consistent triangles each edge will be $T_E = \frac{3*T}{E}$ since each triangle consists of 3 edges. Further T_E must be an integer since there is no way to have a fraction of a triangle. So then $T_E * E = 3 * T$ and so $E \mid 3 * T$.. $\to\leftarrow$

Using triangle analysis we counted the number of consistent or inconsistent triangles (generally inconsistent because they are easier to count) and found whether the number of edges divided three times the number of triangles. This provided a quick test for non-semi-transitivity and helped lead us to some general results.

Our final general method for exploring the possible graphs was to look at the automorphism groups of the orientations in a abstract ways. We used theorems such as the Sylow theorems and Cayley's theorem to genaralize what the automorphism group must look like and contain. Also we studied some subgroups of the undirected graph's automorphism group, and tried to find a subgroup which would provide a S-T orientation of the graph.

5 Complete Graphs with Odd Vertices

The first set of graphs we began to explore were the complete graphs with an odd number of vertices denoted by K_n. A complete graph K_n is a graph with n vertices and every vertex shares an edge with every other vertex. Figure 3, which was denoted as $C_7(1,2,3)$ is also the complete graph of 7 vertices or K_7. We noticed, K_7 has a S-T orientation, defined by the subgroup of U_7 $S = \{1,2,4\}$. So we had one complete graph which was S-T, and this inspired us to look at K_5. After studying circulant orientations of K_5 and not finding any that were S-T, we applied triangle analysis on the graph. We found that K_5 had 5 inconsistent triangles, but it has 10 edges. So obviously 10 does not divide 3*5=15, and so K_5 can not be S-T.

At this point it is important to point out how we studied the inconsistent triangles of the complete graphs. First note that since every vertex is connected to every other vertex, then no matter what two edges you pick, there will always be an edge to form a triangle. Also it is true, since we are only looking for regular orientations, that each vertex has out-degree $\frac{n-1}{2}$ in any orientation. When looking at inconsistent triangles, you can see that any inconsistent triangle has one vertex in which the two edges connected to it are out edges or in edges. Without loss of generality, we chose to look at the out edges. So we know there are $\frac{n-1}{2}$ out edges at each vertex, and any two of these forms a inconsistent triangle, then we know that each vertex is a part of $\binom{\frac{n-1}{2}}{2} = t$ triangles. Then there must be $n * t = T$ inconsistent triangles in any orientation. So for K_5, there are 2 out edges and so $t = 1$, which gives us 5 inconsistent triangles.

The facts we knew about K_5 and K_7 puzzled us. What is so different about 5 and 7. The first important observation we made is that $5 \equiv 1 \mod 4$ and $7 \equiv 3 \mod 4$. This fact led us to the next lemma.

Lemma 5.1 *If $n \equiv 1 \mod 4$ then K_n is not S-T.*

Proof: Suppose $n \equiv 1 \mod 4$, so $n = 4k + 1$. The number of inconsistent triangles at each vertex is

$$t = \binom{\frac{n-1}{2}}{2} = \binom{\frac{4k+1-1}{2}}{2} = \binom{2k}{2} = \frac{2k!}{2!(2k-2)!} = \frac{2k(2k-1)}{2} =$$

$$k(2k-1).$$

So $T = n * k(2k-1)$. The number of edges in K_n

$$E = \binom{n}{2} = \frac{n * 4k}{2} = n * 2k.$$

So

$$\frac{3 * T}{E} = \frac{3 * n * k(2k-1)}{n * 2k} = \frac{3 * k(2k-1)}{2k} = \frac{3(2k-1)}{2}$$

Since 3 and 2k-1 are odd, so the numerator is odd and so 2 does not divide 6k-3 and so E does not divide $3 * T$ and so there can be no S-T orientation of K_n, so K_n is never S-T.

We had now eliminated a substantial portion of our list of possible S-T graphs with this fact. In addition, we were very intrigued about why none of the $n \equiv 1 \bmod 4$ complete graphs were S-T, but we knew that at least one of the $n \equiv 3 \bmod 4$ was S-T. We had already determined through calculations as in the proof above that we could not disqualify any K_n, where $n \equiv 3 \bmod 4$. Therefore we began looking at other complete graphs with $n \equiv 3 \bmod 4$, such as K_{11}, K_{15}, and K_{19}. In examining K_{11} and K_{19}, we found subgroups which provided the necessary jumps and defined a S-T orientation. We then noticed both 11 and 19 like 7 is prime, and using a number theory argument we were able to prove the next statement.

Lemma 5.2 *If n is prime and $n \equiv 3 \bmod 4$, then K_n is S-T.*

Proof: First note that every K_n is defined by $\frac{n-1}{2}$ jumps. Recall from number theory the idea of quadratic residues and quadratic reciprocity modulo a prime p. So if we let $n = p$, then remember that there are $\frac{n-1}{2}$ distinct quadratic residues. Also since if $a \equiv b^2$ and $c \equiv d^2 \bmod$ n, then obviously $ac \equiv (bc)^2$ and $a^{-1} \equiv b^{-2}$ mod n, meaning that the set $S = \{a_1, ... a_{\frac{n-1}{2}} \mid a_i \equiv b^2 \bmod n \text{ for some } b \in U_n\}$ is a subgroup of U_n. Further it is a fact that -1 is not a quadratic residue for $n \equiv 3 \bmod 4$ [Ste52, 143]. Therefore we have a subgroup S of U_n, of size $\frac{n-1}{2}$ and such that $-1 \notin S$, so by our theorem above $\omega = C_n[S]$ is a S-T orient ation of K_n and so K_n is a S-T graph.

These two facts had allowed us to classify a number of possible graphs from our list, and gave us hope that maybe we could show that K_n is S-T only when $n \equiv 3 \bmod 4$ whether or not it is prime. To begin on this study, we began exploring the only other K_n with less than 20 vertices we had no hard evidence on, K_{15}. We struggled with this graph for quite a while. We tried the plug and chug method by looking at every possible circulant orientation (of which there are only 16 thanks to the Adam Isomorphism theorem, a list is given in the appendix). When this did not work, we tried to make up some noncirculant orientations and also tried finding an orientation by looking for subgroups of the undirected graph's automorphism group. Finally we determined that if K_{15} had a S-T orientation, then it must be a circulant orientation [S. Wilson private conversation]. Using *Groups and Graphs* we looked at all 16 possibilities and none of them worked. So we concluded K_{15} is not S-T.

6 Depleted Complete Graphs

After classifying all complete graphs, we turned our attention to a family of circulant graphs known as the depleted complete graphs. A depleted complete graph DK_n is a graph with n vertices and a vertex v is connected to every other vertex except for the vertex $v + \frac{n}{2}$. Every depleted complete graph has an even number of vertices since there is no $\frac{n}{2}$ jump in a graph with an odd number of vertices.

The first group of depleted complete graphs that we looked at was DK_n where $n \equiv 2 \bmod 4$. We already knew DK_6 was S-T since it is the wreath graph $W(3,2)$ and we had found a circulant S-T orientation for DK_{14}. We therefore knew some DK_n graphs were S-T and began looking for some general cases. We began looking at these graphs using the concept of mates. Two vertices are mates if there is no edge between them. What we realized is that if we pair these mates together then we get a two-four copy of $K_{\frac{n}{2}}$. A two-four copy means for every vertex in $K_{\frac{n}{2}}$ there are two vertices in DK_n and for every edge of $K_{\frac{n}{2}}$ there are four edges in DK_n. When we arranged DK_6 and DK_14 into mates we noticed their S-T orientations in this structure mirrored the S-T orientations of K_3 and K_7. This fact lead us to the following:

Theorem 6.1 *DK_n where $n \equiv 2$ mod 4 is S-T if and only if $K_{\frac{n}{2}}$ is S-T.*

> **Proof:** Let $n \equiv 2 \bmod 4$ and let a and b be vertices of DK_n and a' and b' be the mates of a and b respectively. We will look at the out-edges of a in any regular orientation of DK_n. There are $\frac{n-2}{2}$ out-edges, or if $n = 4K + 2$ then there are $2K$ out-edges. So one of two cases can occur. Case 1: If a points to b then b' points to a. If b' points to a then for every vertex c where a points to c, c' points to a (see figure 8). Then there exists an edge between every pair of vertices a points to. Further, this happens at every vertex, and so we can apply triangle analysis on the orientation. There are $2K$ out-edges at every vertex, so each vertex has $\binom{2K}{2} = K(2K-1)$ inconsistent triangles it is a part of, where it is the vertex with two out-edges. Then the total number of inconsistent triangles is $T = n(K(2K-1))$.

The number of edges is $E = \left(\begin{array}{c} n \\ 2 \end{array} \right) - \frac{n}{2} = n(2K)$. So then we have the following:

$$\frac{3T}{E} = \frac{3n(K(2K-1))}{n(2K)} = \frac{6K-3}{2}.$$

This shows that E does not divide 3T and therefore there will be no S-T orientation of this kind. Case 2: If a points to b then a points to b'. So this means that a points to mates, meaning for every c that a points to, then a also points to c'. Suppose d points to a, since if a pointed to d' this would imply that a poined to d, which can not happen. So d' points to a as well. This tells us that when a points to b then a' points to b and a' points to b'. Then we know a set of mates points to another set of mates (see figure). Triangle analysis does not allow us to rule this case out, so therefore this is the only possibility for a S-T orientation of DK_n where $n \equiv 2 \bmod 4$. When we arrange the vertices in mates we then get a two-four copy of $K_{\frac{n}{2}}$. Suppose we had a S-T orientation of DK_n,then when we arrange the vertices as mates to form the two-four copy of the smaller complete graph, and we know mate pairs point in one direction to another mate pair we can use the S-T orientation of DK_n and form a S-T orientation of $K_{\frac{n}{2}}$. We will obtain the orientation by making mates act as one vertex and all four edges act as one edge. Since the big graph is S-T then the new way of thinking of the graph must be S-T and therefore $K_{\frac{n}{2}}$ must have a S-T orientation and so must be S-T. Therefore we have shown that if DK_n is S-T then $K_{\frac{n}{2}}$ is S-T. We will now show the converse. Suppose $K_{\frac{n}{2}}$ has a S-T orientation and let us direct the edges of DK_n arranged as $K_{\frac{n}{2}}$ in this S-T orientation. So if in the smaller complete graph a pointed to b so now a points to b' and a' points to b and b'. Let σ_k be the permutation which takes the edge (a, b) to the edge (c, d). So $a\sigma_k = c$ and $b\sigma_k = d$. Now let σ be a permutation which acts like σ_k, but also has the property that $a'\sigma = c'$ and $b'\sigma = d'$. So σ sends the edge (a, b) to the edge (c, d) and σ sends (a', b') to (c', d'), and all other combinations. So σ defined in such a manner sends each bunch of edges to every other bunch of edges. Now let ρ be a permutation which switches a and a', so $\rho = (aa')$. So ρ is obviously an automorphism of the orientation and ρ will send (a, b) to (a', b). Let τ be a permutation such that $\tau = (bb')$. Again, τ is an automorphism of the orientation and τ sends (a, b) to (a, b'). So ρ and τ will send an edge in a bunch to all other edges in that bunch, and σ sends every bunch to every other bunch, so this orientation is E-T and therfore DK_n is S-T.

Thus, using this fact, we were able to see that DK_{10} and DK_{18} are not S-T and gave us an explanation as to why DK_6 and DK_{14} are S-T. Having classifed all DK_n where $n \equiv 2 \bmod 4$, we began to look at all DK_n where

$n \equiv 0 \bmod 4$. Our smallest example is DK_8. We began exploring this graph using all our previous methods that would apply, but found no S-T orientation. Then while playing with subgroups of the undirected graph we came across an unusual subgroup that provided a S-T orientation for the graph. The subgroup was generated by the two cycles (1256)(3478) and (124)(568). The orientation we found with the subgroup was simplified by switching and renumbering the vertices 1 and 5. The simplified orientation is given in Figure 9.

We began studying why this orientation worked and if it could be applied to other graphs. As a result of our studies we developed the concept of *bendability* and *bendableorientations* (S. Wilson private conversation) which allowed us to come up with a S-T orientation of DK_{16}. The orientation we found is the most unusual S-T orientation we have found thus far. The S-T orientation of DK_{16} is shown in Figure 10. Notice this orientation does have a pattern to it, the 1,4, and 7 jumps are all circulant. The 2 jumps are almost all positive except the edges (10, 8) and (2, 16). The 3 jumps are almost all negative except for the edges (1, 4), (6, 9), (9, 12), and (14, 1). Similarly the 5 jumps are almost all negative except for the edges (1, 6), (4, 9), (9, 14), and (12, 1). Finally, the 6 jumps are almost all positive except for the edges (8, 2) and (16, 10).

Unfortunately the method that allowed us to find the orientation for DK_{16} has yet to allow us to find S-T orientations for DK_{12} and DK_{20}. We strongly feel that they are not S-T, but until we have full explored the concept of bendablility we will not know for sure.

7 $K_{2n,2n,2n,2n}$

A graph $K_{n,n,\ldots,n}$ means a graph with m bunches where m is the number of $n's$ and n vertices in each bunch. Vertices are bunched together if they are mates, and each vertex is connected to every other vertex in every other bunch except in its own bunch. Particularly we focused on the graph $C_{16}(1,2,3,5,6,7)$ which is isomorphic to $K_{4,4,4,4}$. We noticed that DK_8 which is S-T is also $K_{2,2,2,2}$ and therefore we tried using the orientation of DK_8 to derive an orientation for $K_{4,4,4,4}$. We did this be keeping the 1,2, and 3 jumps the same, but we noticed that $5 \equiv -3 \bmod 8$, $6 \equiv -2 \bmod 8$ and $7 \equiv -1 \bmod 8$. Therefore we made the 5,6, and 7 jumps oriented in the negative direction of the 1,2, and 3 jumps respectively. This gave us a S-T orientation for $K_{4,4,4,4}$ which is shown ~~below~~ in figure 11.

In further research on this subject we again used this method to come up with a S-T orientation of $K_{6,6,6,6}$. This lead us to the following conjecture.

Conjecture 1 *Every graph $K_{2n,2n,2n,2n}$ is S-T with a S-T orientation derived from the S-T orientation of $K_{2,2,2,2}$*

We are still working on the proof of this conjecture, but strongly feel it to be true.

8 Orientations Dependent on the Number of Vertices

The first graphs of this type we looked at were graphs with a prime number of vertices. Soon after we started studying these graphs we were able to narrow the possible S-T orientations the graphs could have to circulant orientations. We did this is by proving the following lemma.

Lemma 8.1 *A graph with a prime number of vertices is S-T if and only if it has a S-T circulant orientaion.*

> **Proof:** Suppose Γ has a prime number of vertices p and a S-T orientation, ω. So ω must be V-T and so the orbit of vertices under the action by the automorphism group is of size p. Then the Orbit-Stabalizer theorem tells us that $|G = Aut(\omega)| = p * |S_G(v)|$ where v is any vertex, this means then that $p \mid |Aut(\omega)|$[Fra67, 155]. Cayley's theorem then tells us that $Aut(\omega)$ has an element of order p, which in our case must be a p-cycle [Fra67, 72]. Meaning then that by lemma 1.2 ω must be circluant. If Γ has a prime number of vertices and is S-T then the S-T orientation is circulant. The converse is true since by definition if Γ has a S-T circulant orientation, and therefore Γ is S-t.

Using this fact we studied and eliminated all the possible circulant orientations of $C_{13}(1,5)$, $C_{17}(1,4)$, and $C_{17}(1,2,4,8)$. A list of the possible circulants is given in the appendix.

The next set of graphs we turned our attention to were graphs with an even number of vertices and jumps $a_1, ..., a_k$ such that a_i is odd for all i. The first of these graphs was $C_{10}(1,3)$. Notice in these graphs, because all the jumps are odd, then no even vertex will share an edge with any other even vertex, and similarly for odd vertices. Using this fact and the fact that 10 must divide the order of the automorphism group of any orientation ω, we knew that either there was a 10-cycle, or a an element made up of two 5-cycles. If there was a 10-cycle, this meant the orientation was circulant, but this possibility was quickly eliminated by the plug and chug method. So we knew that there had to be an element of two 5-cycles for there to be a S-T orientation, and we also knew that we could relabel the vertices, using the permutation $(1\ 3\ 5\ 7\ 9)(2\ 4\ 6\ 8\ 10)$. This means that any jump, which goes out or in from 1, would have to do the same at 3, 5, 7 and 9. The even vertices would have to act similarly. So the only other possible orientation would be where a specific jump would go out one of parity of vertices and into the other. Using one of these orientations, we found a S-T orientation of the graph. The orientation if denoted by $C_{10}[+1-, -3+]$ meaning the 1 jumps point out from evens and the 3 jumps point out from the odds. A picture is given in Figure 12.

Using this orientation and the fact about graphs with even number of vertices and realitively prime jumps we were also able to come up with new S-T orientations for the graphs $C_{12}(1,5)$ and $C_{20}(1,3,7,9)$.

When looking at our list of S-T circulants thus far, the graph $C_{10}(1,3)$ was one of the oddest we had. It had no family we could put it in. So we studied the graph to see what sort of properties it had. One of the properties of the orientation we found was that since gcd(3,10)=1, then we could generate a automorphism which would send the 3 jumps and make them act as one jumps. The automorphism we used was to send the 0 vertex to the 1 vertex and then arranged the vertices around in 3's. The final permutation looked like $\sigma = (0187)(2563)(49)$. So we began to look at what allowed such an automorphism to exist. We first realized that 3 and 10 were relatively prime, but more than that, we saw that $3^2 \equiv -1 \bmod 10$. So we took the next odd number, 5, and looked at $C_{26=5^2+1}(1,5)$, gave it a similar orientation to our $C_{10}(1,3)$ orientation and found that it was S-T. So we were then able to generalize it into the following fact.

Theorem 8.1 *If a is odd, $N = a^2+1$, then $C_N(1,a)$ is S-T, with an orientation $\omega = C_N[+1-,-a+]$.*

> **Proof:** Suppose a is odd and $n = a^2+1$ and let ω be the orientation described above. Let $\sigma = (135...n-1)(024...n-2)$ when we apply this permutation it is obviously an automorphism and it sends a positive 1 jump to every other positive 1 jump. Let $\tau = (02)(n-13)...(\frac{n}{2}-1\frac{n}{2}+1)$. When we apply τ this sends the positive 1 jumps to its negative 1 jump counterpart. By these permutations we know that all the 1 jumps and by the same argument all the 3 jumps are similar. Let $\rho = f(x)$ where $f(x) = ax - a$. This permutation sends 0 to 1 and then arranges the remaining vertices clockwise so that for a vertex v the vertex clockwise to it is the vertex $v + a$. Further σ changes the parity of each vertex so that even vertices must now act as the odd vertices did before the permutation and vice-versa. It is apparent that we have sent the 3 jumps to where the 1 jumps were before and therefore must now act as the 1 jumps did before the permutation. Since in the original graph the 3 jumps at the even vertices acted like the 1 jumps at the odd vertices then by sending the 3 jumps to the 1 jumps and changing the parity of the vertices, thus, the 3 jumps are again oriented correctly after the permutation. We must ensure that the 1 jumps have now moved to the 3 jumps. Since on the outside each vertex moves 3 steps, so the a vertex away from any vertex is now the a^2 away vertex. We know $a^2 \equiv -1$ mod n, and therefore this vertex is also the $v-1$ vertex of the original graph. Then there must exist an edge there which must be the original 1 jumps and by a similar argument from above these jumps are also oriented correctly. Thus, the 1 jumps moved to the 3 jumps, and σ is an automorphism of the graph. So by applying σ we can move an a edge into a 1 edge and thus will move all the edges around. Therefore the orientation is E-T and the graph is S-T.

Next we turned our attention to the graph $C_{18}(1,3,5,7)$ thinking we could use the same method to find a S-T orientation of the graph. We soon realized that the 3 jump was a problem because it was not realitively prime to 18. We noticed that in order to keep the graph regular, a S-T orientation would have to have all circulant jumps, all jumps like those of $C_{10}(1,3)$, or two circulant jumps and two like those in $C_{10}(1,3)$. But similarly to the argument for the graph of $C_{10}(1,3)$ we know $C_{18}(1,3,5,7)$ must have a 18, 9, 6, 3, or 2-cycle. If it had an 18-cycle then the S-T orientation would be circulant, but we tried all the circulant possibilities and none were S-T. If it had a 9-cycle then the graph would have an orientation with either two jumps acting as those in $C_{10}(1,3)$ or four jumps and we have also exhausted these as possibilities. Therefore if this graph does have a S-T orientation it will be one we have not yet seen.

Further study into the graph $C_{18}(1,3,5,7)$ reveals that it is a part of the family of graphs known as $DK_{n,n}$. These graphs have two sets of n vertices which do not share an edge, and every vertex in a bunch shares an edge with all but one vertex in the opposite bunch. When n is odd, we know that $DK_{n,n}$ is isomorphic to $C_{2n}(1,3,5,...,n-2)$. So the jumps are all odd numbers up to, but not including n, meaning there are $\frac{n-1}{2}$ jumps. We will not deal with the case when n is even, since the degree of a vertex would be odd, and so the graph would not be regular. Thus, $C_{18}(1,3,5,7)$ is $DK_{9,9}$. Other S-T graphs of this type are, $C_{10}(1,3)$ is $DK_{5,5}$, and $C_{14}(1,3,5)$ is $DK_{7,7}$. Both of these have two orientations the first being the orientation we have explained in the $DK_{5,5}$ case, and the second orientation being a subgroupXS. We then generalized our thoughts and were able to prove the next fact.

Theorem 8.2 *If p is prime and $p \equiv 3 \bmod 4$, then $DK_{p,p}$ is S-T.*

> **Proof:** Suppose p is prime and $p \equiv 3 \bmod 4$. So $DK_{p,p}$ is $C_{2p}(1,3,5,...,p-2)$. Since p is a prime, then 2p has a primitive root. Further since $\phi(2p) = p-1$ then there are $\frac{p-1}{2}$ quadratic residues mod 2p, and -1 is not a quadratic residue. So the subgroup of quadratic residues has the correct size we need and does not include -1, but can we be sure the right jumps are in the subgroup. Let a be one of the undirected jumps. Either a is a quadratic residue, meaning a is in our subgroup and we are fine, or a is not a quadratic residue. Suppose a is not a quadratic residue, so $a \equiv g^{2m+1}$ mod 2p. But since we know $-1 \equiv g^{2k+1}$ mod 2p, so $-a \equiv g^{2k+1}g^{2m+1} \equiv g^{2k+2m+2} \equiv g^{2(k+m+1)}$ mod 2p. So either a or -a is in our subgroup S, the subgroup of quadratic residues, and never both. So then $C_{2p}[S]$ is a S-T orientation of $DK_{p,p}$, and so $DK_{p,p}$ is S-T.

Also we feel strongly that soon we will be able to prove the next conjecture.

Conjecture 2 *If p is prime and $p \equiv 1 \bmod 4$, then $DK_{p,p}$ is S-T.*

The proof of this will rely more on permutations, and the relationships of relatively prime numbers. The orientation for these graphs will be something like the orientation of $C_{10}(1,3)$. We are still looking for ways to prove things about $DK_{n,n}$ when n is odd and composite.

9 K_5 is Wonderful

A wise man named Steve Wilson once said, "Sometimes in math, your hardest smallest case can be used to give you answers to larger problems." We have found this statement to be very true for some of our larger graphs, in which we have used the graph of K_5 to eliminate them from our list of possible S-T graphs. The first case is the graph $C_{15}(1,2,3,4,6,7)$ also known as the graph $K_{3,3,3,3,3}$.

We have found the graph $C_{15}(1,2,3,4,6,7)$ not to be S-T. This is because the graph must have regular degree then each vertex has 6 in and 6 out edges. Let the bunches be called 0,1,2,3,4. If there is a vertex in bunch 0 that has an out edge to one and only one vertex in bunch 1 then that would leave 5 other out edges to point out to the 3 remaining bunches. But 3 does not divde 5 and therefore we would two bunches with two in edges from bunch 0 and two bunches with 1 in edge from bunch zero. Without losing generality assume bunch 0 has 1 out edge to bunches 1 and 2, and 2 out edges to bunches 3 and 4. Then we could never find an automorphism from bunch 1 to bunch 3 since they are not similar and therefore an orientation of this kind could never be S-T. A similar argument tells us that a vertex from bunch 0 could not have two and only two out edges to two vertices in bunch 1 because this would leave 4 out edges to be divided among 3 bunches. Therefore if a vertex in bunch 0 points to a vertex in bunch 1 it must point to all vertices of bunch 1. Meaning the vertex in bunch 0 will point to all vertices in exactly two bunches, assume these are bunches 1 and 2 and so does not point to the vertices in bunches 3 and 4. Therefore all the vertices in bunches 3 and 4 must point to all the vertices in bunch 0. So by our argument above this means that each vertex in one of these bunches points to every vertex in bunch 0 and so this happens at every bunch. Thus, $K_{3,3,3,3,3}$ becomes a three-nine copy of K_5 and therefore if there was a S-T orientation of the copy there would be a S-T orientation of K_5. We know K_5 has no such orientation and therefore $C_{15}(1,2,3,4,6,7)$ is not S-T.

The next graph we looked at was $C_{20}(1,2,3,6,7,9)$ which we found to be isomorphic to $K_4 \times K_5$. A cross product of graphs, $\Gamma_1 \times \Gamma_2$, is a graph whose vertices are ordered pairs (v,w) where v is a vertex of Γ_1 and w is a vertex of Γ_2. With edges $\{(v,w),(y,z)\}$ if $\{v,y\}$ and $\{w,z\}$ are edges of Γ_1 and Γ_2 respectively. If you arrange the vertices in 5 groups of 4 with the bunches being {1,6,11,16}, {2,7,12,17}, . . . then this arrangement is similar to K_5. Between bunches every vertex is connected to every other vertex except for the vertex which is a jump of a multiple of 4. So 1 and 17 are not connected and 2 and 14 are not connected. Between bunches there are 3 edges at each vertex and from a similar argument above we know any possible S-T orientation will be arranged such that all edges from bunch a to bunch b will be oriented in the same direction. So this graph can not be S-T by the same K_5 argument as above.

The last graph in this group is $C_{20}(1,2,3,4,6,7,8,9)$ which is also $K_{4,4,4,4,4}$. Through much analysis we have many restrictions on possible orientations, but as of yet no definite conclusions about them.

10 Conclusion

In conclusion, we have studied all E-T graphs with the number of vertices between 5 and 20. We have determined whether or not 45 out of the original 49 graphs were S-T. To those graphs we have found to be S-T we have assigned them to one of the following families: subgoups, wreath graphs, depleted wreath graphs, complete graphs with a prime number of vertices, p where $p \equiv 3 \bmod 4$, Depleted complet graphs where the number of vertices, n is congruent to 2 mod 4 and the graph $K_{\frac{n}{2}}$ is S-T, Bendable graphs, $K_{2n,2n,2n,2n}$, and oddities. We have general theorems about these families and general results about these graphs and those with greater than 20 vertices. We still have much to do in completing our studies of the four unknown graphs. In addition, we hope to obtain more general results so that we will be able to tell if any graph is S-T just by studying the number of vertices and jumps of the graph.

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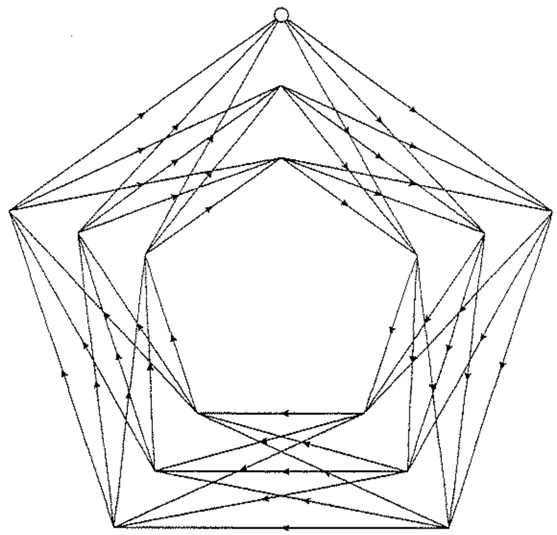

Figure 5: $W(5,3)$

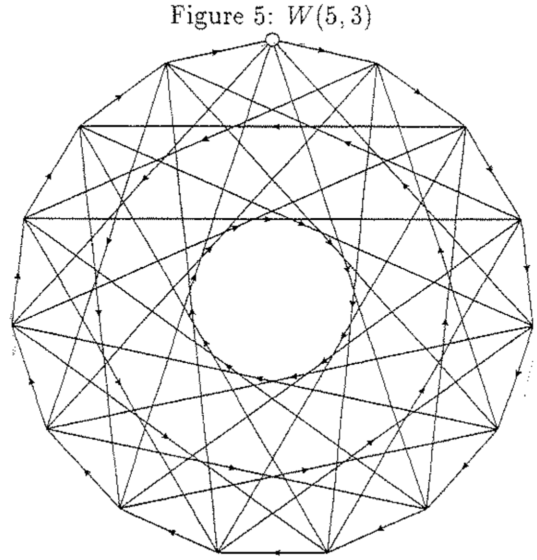

Figure 6: $C_{15}[1,6,11]$

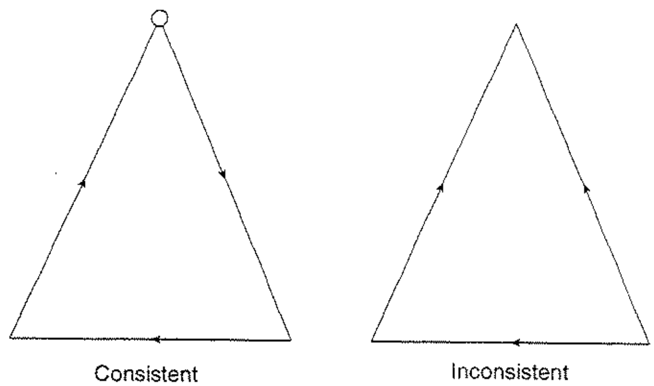

Figure 7:

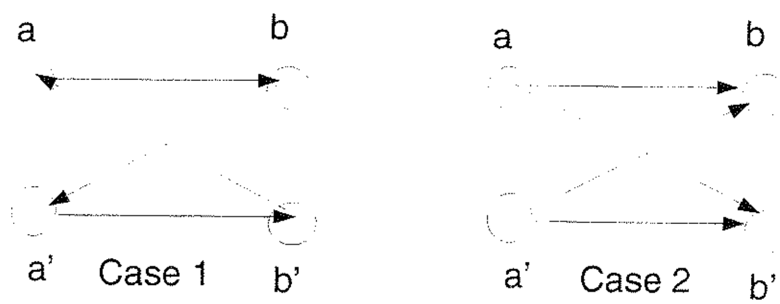

Figure 8:

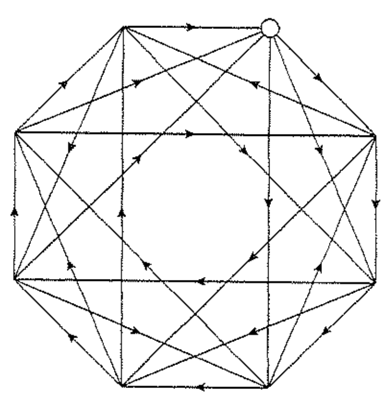

Figure 9: DK_8

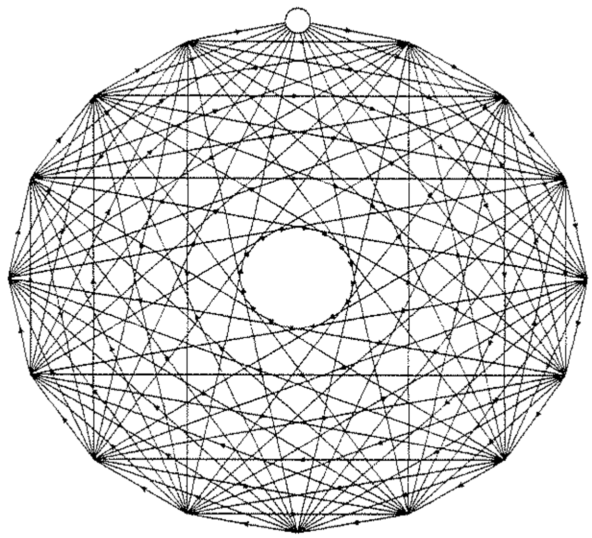

Figure 10: DK_{16}

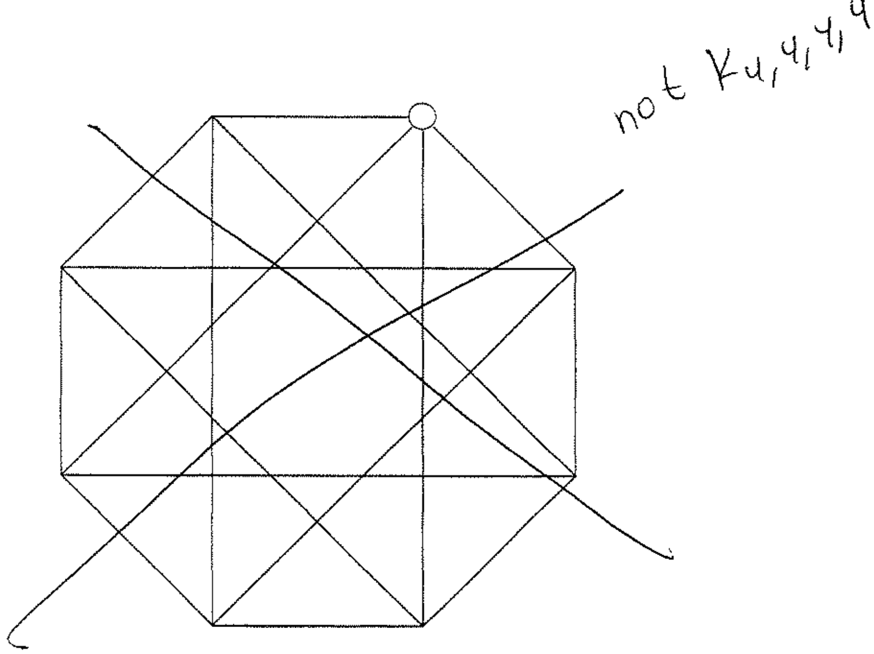

Figure 11: $K_{4,4,4,4}$

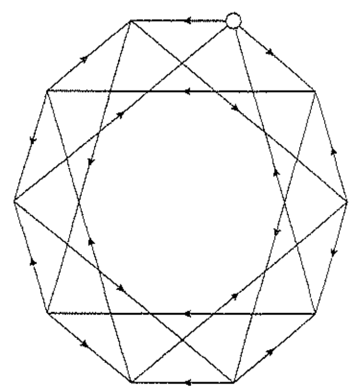

Figure 12: $C_{10}[+1-,-3+]$

Apendix A

E-T Circulants

N	Jumps	E-T	S-T
5	1,2	Y	n
6	1,2	Y	y
7	1,2,3	Y	y
8	1,3	Y	y
8	1,2,3	Y	y
9	1,2,4	Y	y
9	1,2,3,4	Y	n
10	1,3	Y	y
10	1,4	Y	y
10	1,2,3,4	Y	n
11	1,2,3,4,5	Y	y
12	1,5	Y	y
12	2,4	Y	d
12	1,3,5	Y	y
12	1,2,4,5	Y	y
12	1,2,3,4,5	Y	
13	1,5	Y	n
13	1,3,4	Y	y
13	1,2,3,4,5,6	Y	n
14	1,6	Y	y
14	1,3,5	Y	y
14	2,4,6	Y	d
14	1,2,3,4,5,6	Y	y
15	1,4	Y	y
15	3,6	Y	d
15	1,4,6	Y	y
15	1,2,4,7	Y	y
15	1,2,4,5,7	Y	y
15	1,2,3,4,6,7	Y	n
15	1,2,3,4,5,6,7	Y	n
16	1,7	Y	y
16	2,6	Y	d
16	2,4,6	Y	d
16	1,3,5,7	Y	y
16	1,2,3,5,6,7	Y	
16	1,2,3,4,5,6,7	Y	
17	1,4	Y	n
17	1,2,4,8	Y	n
17	1,2,3,4,5,6,7,8	Y	n
18	1,8	Y	y
18	3,6	Y	d
18	1,5,7	Y	y
18	2,4,8	Y	d
18	1,3,5,7	Y	
18	2,4,6,8	Y	d

18	1,2,4,5,7,8	Y	y
18	1,2,3,4,5,6,7,8	Y	n
19	1,7,8	Y	y
19	1,2,3,4,5,6,7,8,9	Y	y
20	1,9	Y	y
20	2,6	Y	d
20	4,8	Y	d
20	1,6,9	Y	y
20	1,3,7,9	Y	y
20	1,4,6,9	Y	y
20	2,4,6,8	Y	d
20	1,3,5,7,9	Y	y
20	1,2,3,6,7,9	Y	
20	1,2,3,4,6,7,8,9	Y	
20	1,2,3,4,5,6,7,8,9	Y	

Apendix B

S-T Circulants

N	Jumps	Orientation	# Edges	|Aut(G)|	|Aut(O)|	|H|	Notes
5	1,2	None	10	120			5=1mod4
6	1,2	1,4	12	48	24	2	W(3,2)
7	1,2,3	1,2,4	21	5040	21	7	Subgroup and 7=3mod4
8	1,3	1,3={+1-,-3+}	16	1152	16	4	Subgroup
8	1,3	1,5	16	1152	64	2	Subgroup
8	1,2,3	1,{2,-2},3	24	384	24	8	Bendable
9	1,2,4	1,4,7	27	1296	648	3	W(3,3)
9	1,2,3,4	None	36	362880			9=1mod4
10	1,3	{+1-,-3+}	20	240	20	5	even vertices, odd jumps
10	1,4	1,6	20	320	160	2	W(5,2)
10	1,2,3,4	None	40	3840			triangle analysis
11	1,2,3,4,5	1,3,4,5,9	55	39916800	55	11	Subgroup and 11=3mod4
12	1,5	1,5	24	768	24	3	Subgroup and W(6,2)
12	1,5	1,7	24	768	384	2	Subgroup
12	1,5	{+1-,-5+}	24	768	24	6	even vertices, odd jumps
12	1,3,5	1,5,9	36	1086800	5184	3	W(4,3)
12	1,2,4,5	1,4,7,10	48	82944	41472	4	W(3,4)
12	1,2,3,4,5		60	46080			
13	1,5	None	26	52			must have circulant orientation
13	1,3,4	1,3,9	39	78	39	13	Subgroup
13	1,2,3,4,5,6	None	78	13!			13=1mod4
14	1,6	1,8	28	1792	896	2	W(7,2)
14	1,3,5	1,9,11	42	10080	42	7	Subgroup
14	1,2,3,4,5,6	1,2,4,8,9,11	84	645120	2688	14	K_(n/2)=K_7
15	1,4	1,4	30	60	30	5	Subgroup
15	1,4	1,11	30	60	30	3	Subgroup
15	1,4,6	1,6,11	45	77760	38880	3	W(5,3)
15	1,2,4,7	1,2,4,8	60	720	60	15	Subgroup
15	1,2,4,5,7	1,4,7,10,13	75	10368000	5184000	5	W(3,5)
15	1,2,3,4,6,7	None	90	933120			K_5 argument
15	1,2,3,4,5,6,7	None	105	15!			must have circulant orientation
16	1,7	1,7={+1-,-7+}	32	4096	32	8	Subgroup and W(8,2)

16	1,7	1,9	32	4096	2048	2	Subgroup
16	1,3,5,7	1,5,9,13	64	2^15*3^4*25*49	1327104	4	Subgroup and W(4,4)
16	1,3,5,7	1,3,9,11={+1-,-3+,-5+,+7-}	64	2^15*3^4*25*49	4096	8	Subgroup
16	1,2,3,5,6,7	1,{2,-2},3,{-6,6},9,11	96	7962624	6144	16	from C_8(1,2,3)
16	1,2,3,4,5,6,7	1,{2/(8-6,16-14)}	112	10321920	336	16	Bendable
		{-3/(1-4,8-11,9-12,					
		16-3)},4,{-5/(1-6,					
		8-13,9-14,16-5)},					
		{6/(12-2,4-10)},7					
17	1,4	None	34	68			must have circulant orientation
17	1,2,4,8	None	68	136			must have circulant orientation
17	1,2,3,4,5,6,7,8	None	136	17!			17=1mod4
18	1,8	1,10	36	9216	4608	2	W(9,2)
18	1,5,7	1,7,13	54	559872	279936	3	W(6,3)
18	1,3,5,7		72	725760			
18	1,2,4,5,7,8	1,4,7,10,13,16	108	2^13*3^7*125	2^12*3^7*125	6	W(3,6)
18	1,2,3,4,5,6,7,8	None	144	185794560			triangle analysis
19	1,7,8	1,7,11	57	114	57	19	Subgroup
19	1,2,3,4,5,6,7,8,9	1,4,5,6,7,9,11,16,17	171	19!	171	19	Subgroup and 19=3mod4
20	1,9	1,9={+1-,-9+}	40	20480	40	10	Subgroup and W(10,2)
20	1,9	1,11	40	20480	10240	2	Subgroup
20	1,6,9	1,6,11	60	240	120	4	DW(5,4)
20	1,3,7,9	1,3,7,9	80	245760	80	10	Subgroup
20	1,3,7,9	1,9,13,17	80	245760	480	5	Subgroup
20	1,3,7,9	{+1-,-3+,-7+,+9-}	80	245760	20480	10	even vertices, odd jumps
20	1,4,6,9	1,6,11,16	80	79626240	39813120	4	W(5,4)
20	1,3,5,7,9	1,5,9,13,17	100	2^17*3^8*5^4*49	2^14*3^4*5^4	5	W(4,5)
20	1,2,3,6,7,9	None	120	2880			K_5 argrument
20	1,2,3,4,6,7,8,9		160	955514880			
20	1,2,3,4,5,6,7,8,9		180	2^18*3^4*25*7			