The vertex next door:

a look at common neighbors and other matters regarding p-graphs

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0.1 Introduction

P-graphs are connected, bi-partite graphs with vertices labelled as subsets of the natural numbers and edges defined through containment of these subsets. This defining element of set containment makes the p-graphs and wedge products of p-graphs very well suited to combinatorial analysis. We are especially concerned with the notion of neighbors and common neighbors of vertices in these p-graphs, and with providing a summation of information about the p-graphs in vector format.

0.2 Acknowledgements

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0.3 Notes on background

Background materials and knowledge provided in this paper and used in the course of the REU were provided by Professor Wilson through lectures and handouts.

0.4 Basic assumptions about graphs

A *graph* is a set of edges attached to a set of vertices. For our purposes, all graphs will be connected and have no loops or parallel edges. Our graphs will be **bi-partite**, that is, two-colorable, with every edge connecting a white vertex to a black vertex (insert pictures).

A **symmetry** or **automorphism** of a graph Γ is a permutation of its vertices which preserves edges (insert pictorial example). $Aut(\Gamma)$ is the collection of all symmetries of Γ, and it is a group under composition.

A **dart** is a directed edge of Γ, the ordered pair (A, B) or (B, A) from the edge {A,B} connecting vertices A and B (insert picture). If we consider a subgroup G of $Aut(\Gamma)$ which is transitive on edges of Γ but not vertices, then the orbit of a dart under G consists of one dart from each edge. We may consider this orbit as a directed graph. $Aut^+(\Gamma)$ is the group of color-preserving symmetries of Γ, transitive on black vertices and transitive on white vertices. Hence, all of the white vertices are in one orbit and must have the same degree d, and all black vertices are in the other orbit and have the same degree e. From the bi-partite condition and the above, we say that Γ is **bi-transitive**. Furthermore, we say Γ is **strictly bi-transitive** provided that $Aut^+(\Gamma) = Aut(\Gamma)$; that is, if the graph does not have a symmetry which reverses color.

We call our graphs **regular** if all vertices have the same degree, that is, if d=e, and if we have equal number of black and white vertices. Results presented here are not dependent on regularity.

0.5 A very special sort of graph: p-graphs

Our investigations center around a particular type of bi-partite graphs called *p-graphs.*

For the set $X = \{1, 2, 3, \dots, n\} \subseteq \mathbb{N}$, a **p-graph**, notated $P_n(a, b)$, defined for integers $0 < a < b < n$, has two kinds of vertices:

- those corresponding to subsets of X of size b, which we will call **black**
- those corresponding to subsets of X of size a which we will call **white**

A set A of size a is connected to a set B of size b when $A \subset B$.

(insert picture)

As the edges are defined by set containment, the degree of a black vertex is $\binom{b}{a}$ and the degree of a white vertex is $\binom{N-a}{b-a}$.

$P_n(a, b)$ is bi-transitive for all a, b, n and strictly bi-transitive unless $a+b = n$

0.6 The Vertex Next Door

Most of our questions surrounding p-graphs center around the idea of "common neighbors."

A **neighbor** of a vertex A is simply a vertex, B, of the other color, which A is connected to by an edge; in the case of p-graphs this implies that either $A \subset B$ or $B \subset A$ (insert picture). It is important to note that p-graphs are **worthy**, i.e., that no two distinct vertices have the same set of neighbors.

We consider two vertices to **share a common neighbor** or be **neighborly** if they both connect to a given vertex of the opposite color (insert picture).

0.7 When do common neighbors exist?

Brief contemplation of p-graphs tells us that two black vertices (of size b) will share a common neighbor if they contain a common subset of size a or larger; similarly, two white subsets (of size a) will share a common neighbor if they are both contained in a set of size b. We developed two propositions to state this more formally:

Proposition 1 *Consider $P_n(a,b), b > a$; every two vertices corresponding to sets of size b must share a common neighbor of size a if and only if $a \leq 2b-n$.*

Proof of Proposition 1 *Consider subsets of the natural numbers, $S = \{1,\dots,n\}$, of size $a \leq 2b-n$ and b. Consider two subsets of size b such that set $B_1 = \{1,\dots,b\}$ and set $B_2 = \{(n-b)+1,\dots,n\}$; by exploiting the ordering on $\{1,\dots,n\}$, we know these subsets will be the two which definitely will express the "most separated" subsets of $\{1,\dots,n\}$; that is, the subsets with the least possible intersection. From this construction of B_1 and B_2 we see that the size of the intersection of these two sets will be $2b-n$. Now, if $a \leq 2b-n$ then B_1 and B_2 must share a subset of size a in either their intersection or a subset of their intersection. Hence B_1 and B_2 will be neighborly via a subset of size a.*

Now consider two sets of size b, B_1 and B_2, which have a common neighbor A of size a; that is, a shared subset of size a. Since A contains elements which are in both B_1 and B_2, $A \subseteq B_1 \cap B_2$. Considering our "worst-case" for B_1 and B_2 as described above, the intersection of the two is of size $2b-n$; hence $a \leq 2b-n$.

Proposition 2 *Consider* $P_n(a,b), b > a$; *every two vertices corresponding to sets of size* a *share a common neighbor of size* b *if and only if* $a \leq \frac{b}{2}$.

Proof of Proposition 2 *Consider two sets,* A_1 *and* A_2 *of size* $a \leq \frac{b}{2}$. *In our "worst case," we have* $A_1 \cap A_2 = \emptyset$, *in which case* $A_1 \cup A_2 = 2a$. *Since* $a \leq \frac{b}{2}$, *we know that* $2a \leq b$; *hence, we can construct a set* B *of size* b *which contains* $A_1 \cup A_2$ *and which will thus be their common neighbor.*

Now consider two sets of size a, A_1 *and* A_2, *which have a common neighbor* B *of size* b; *that is, a set which contains both* A_1 *and* A_2. *Since* B *contains all the elements in both* A_1 *and* A_2, $A_1 \cup A_2 \subseteq B$. *Considering our "worst case,"* $A_1 \cap A_2 = \emptyset$, b *would necessarily have to be greater than* $2a$, *i.e.* $a \leq \frac{b}{2}$.

0.8 The Ivanov vector

In a 1987 paper, Ivanov[1] developed a set of conditions and identities for certain properties of graphs with given numbers of vertices, etc., including defining a vector which encodes information about common neighbors of vertices. We interpret and expand his definition to apply to p-graphs as follows:

We define two vectors for each $P_n(a,b)$:

- the white vector whose length is the degree of the white vertices plus one, that is $d+1$, with the elements of the vector being indexed by 0 through d.

- the black vector whose length is the degree of the black vertices plue one, that is $e+1$, with the elements of the vector being indexed by 0 through e.

These vectors each are calculated using a single, base vertex of their color, when in conjunction with other representitive vertices provides information for all vertices of that color, a process which will be explained in the next section.

The i^{th} entry in the vector indicates the number of vertices which share i common neighbors with the base vertex.

[1] Ivanov, A.V.. "On edge but not vertex transitive regular graphs." *Annals of Discrete Mathematics 34*. 1987. 273-286.

(insert picture)

Ivanov vectors are important in our discussions of strict bitransitivity, as a graph is strictly bitransitive if its black and white vectors are not identical.

0.9 Building the Ivanov vector

The following two propositions allow us to build the Ivanov vector for a given p-graph with relatively little pain and suffering. Part 1 of each proposition gives the index i for the vector, and Part 2 gives the entries to be placed in a given index. It is important to note that in order to build the complete Ivanov vector, we must perform the calculations in these two propositions for each possible value of x, the value of the intersection with the base vertices B_1 and A_1, in the p-graph.

Proposition 3 *Consider $P_N(a,b)$, $b > a$. Given two sets of size b, B_1 and B_2 with $|B_1 \cap B_2| = x$ and $|B_1 \cup B_2| = y$, then (1.) the number of common neighbors shared by B_1 and B_2 is $\binom{x}{a} = i$ and (2.) the number of B_i's, including B_2, which share i common neighbors with B_1 is*

$$\binom{b}{x}\binom{N-b}{b-x}.$$

Proof of Proposition 3 *(1.) We know that B_1 and B_2 share a common neighbor A if A is a subset of both of them; that is, if it is contained in their intersection. Thus we may find the number of common neighbors by the given binomial coefficient.*
(2.) Since our relationship of neighborliness is based on set containment, we need only build B_i's which also intersect B_1 in x elements in order to build the sets which share i common neighbors, the $\binom{b}{x}$ gives us the number of the intersections of size x which are possible from the elements of size b, thus providing starting points for the construction of our B_i's. We complete the B_i's with the $\binom{N-b}{b-x}$ which selects enough elements from those not in B_1 to complete a set of size b already partially filled by the x elements of the intersection. Note that these binomial coefficients are merely dealing in sizes and not telling us which particular elements are selected.

Proposition 4 *Consider $P_N(a,b)$, $b > a$. Given two sets of size a, A_1 and A_2 with $|A_1 \cap A_2| = x$ and $|A_1 \cup A_2| = y$, then (1.) the number of common neighbors shared by A_1 and A_2 is $\binom{N-y}{b-y} = i$ and (2.) the number of A_i's, including A_2, which share i common neighbors with A_1 is*

$$\binom{a}{x}\binom{N-a}{a-x}.$$

Proof of Proposition 4 *(1.) We know that A_1 and A_2 share a common neighbor B if they are both subsets of B; hence $A_1 \cup A_2$ must be contained in B. Thus finding the number of common neighbors of A_1 and A_2 reduces to finding how many sets of size b contain a set of size $|A_1 \cup A_2|$, which is expressed through the given binomial coefficient.*
(2.) Here we wish to determine how many sets A_i, including A_2, we could build to have the same number of common neighbors with A_1 as A_2. From (1.) we know that the number of common neighbors is dependent on the size of the union of A_1 and A_2, which is, in turn, dependent on the size of the intersection of A_1 and A_2. So we wish to build A_i's which have an intersection of size x with A_1. The $\binom{a}{x}$ gives us the number of the intersections of size x which are possible from the elements of size a, thus providing starting points for the construction of our A_i's. The $\binom{N-b}{b-x}$ indicates the number of possibilities for completing the A_i's by selecting $a-x$ elements from the number of those not already used in x. Note that these binomial coefficients are merely dealing in sizes and not particular elements.

We see that both our indices and their entries are strongly dependent on x.

0.10 Calculating the Ivanov vector

As an example for calculating our p-graph Ivanov vector, we will consider the black vector for $P_7(2,6)$ (insert picture).

Using Proposition 3, we first take representitive black vertices in $P_7(2,6)$, namely (123456) and (234567), with (123456) being our base vertex. As $|(123456) \cap (234567)| = 5$, part 1 of Proposition 3 says they have $\binom{5}{2} = 10$

common neighbors. Hence all vertices which have an intersection of 5 with (123456) will be included in the 10th term of the vector. Part 2 of Proposition 3 says then that the 10th term of the vector will be $\binom{6}{5}\binom{1}{1} = 6$. We then must consider the sizes of other possible intersections of the vertices of size 6; as we are working with $n = 7$, every pair of vertices of size 6 will intersect in 5 elements. As there was only one possible value for the intersection of two black vertices of size 6, we know that $i = 6$ will hold the only non-zero entry in the vector. The black vector for $P_7(2,6)$ will then be (0,0,0,0,0,0,0,0,0,6,0,0,0,0,0).

Similarly we use the second proposition to calculate the white Ivanov vector and find it to be (0,0,0,10,10,0).

0.11 The Wedge

We combine two p-graphs by taking their **wedge product**.

In general, if Γ_1 and Γ_2 are graphs, the wedge product $\Gamma_1 \wedge \Gamma_2$ is a graph whose vertices are ordered pairs of vertices from the factors, with (a,b) connected to (c,d) if and only if $\{a,c\}$ is an edge in Γ_1 and $\{b,d\}$ is an edge in Γ_2.

For example, the product $P_5(2,3) \wedge P_5(3,4)$ would contain the edge connecting the vertices $(34, 123)$ and $(234, 1235)$.

In bipartite graphs such as our p-graphs, the wedge product will have two components, one in which $(white, white)$ vertices are connected to $(black, black)$ vertices and one in which $(black, whites)$ are connected to $(white, blacks)$. We consider only the latter.

From the definition of the wedge product, we obtain the following method for determining the common neighbors of points in the wedge vector:

Proposition 5 *Consider $a, c \in \Gamma_1$ and $b, d \in \Gamma_2$. Say (a,b) and (c,d) have common neighbors in $\Gamma_1 \wedge \Gamma_2$, then these common neighbors are of the form (i,j) where i is a common neighbor of a and c in Γ_1 and j is a common neighbor of b and d in Γ_2.*

Proof of Proposition 5 *If (i,j) is a common neighbor of (a,b) and (c,d) in $\Gamma_1 \wedge \Gamma_2$, then there is an edge connecting (a,b) to (i,j) and one connecting (c,d) to (i,j). Because of how we define edges in a wedge-product-generated graph, this indicates that a and c must both be connected to i, and b and d*

must both be connected to j. Hence i is a common neighbor of a and c in Γ_1 and j is a common neighbor of b and d in Γ_2.

Now say we have i which is a common neighbor of a and c in Γ_1 and j which is a common neighbor of b and d in Γ_2. Then we know there are edges connecting i to a and c and j to b and d. When we consider $\Gamma_1 \wedge \Gamma_2$, we then have edges connecting both (a, b) and (c, d) to (i, j), which makes it a common neighbor of the two of them.

For example, consider $P_5(2,3) \wedge P_5(1,4)$. The vertices $(12, 5)$ and $(23, 4)$ have the common neighbor $(123, 1245)$ as (12) and (23) have the common neighbor of (123) in $P_5(2,3)$, and (4) and (5) have the common neighbor (1245) in $P_5(1,4)$.

0.12 Building the Ivanov vector for a wedge product

We first suspected that the Ivanov wedge vector might be entirely multiplicative; however, it is not, primarily because of the need to include common neighbors of vertices which have either the same first or second coordinate in the wedge product. Taking this into consideration, the following two methods provide a system for building the Ivanov wedge vectors:

Method 1 *Given $P_N(a, b)$ and $P_N(c, d)$, with "black" vectors $(\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{\binom{b}{a}})$ and $(\tau_0, \tau_1, \tau_2, \ldots, \tau_{\binom{d}{c}})$ respectively, then the black vector for $P_N(a, b) \wedge P_N(c, d)$ may be found through the following process:*

First of all, we know that we will have $\binom{b}{a}\binom{d}{c}$ black vertices in the wedge product which much hold places in the wedge vector. In examining common neighbors, we first consider those pairs of vertices in the wedge which are of the form (b_1, d_1) and (b_2, d_2) where $b_1 \neq b_2$ and $d_1 \neq d_2$. For these vertices, the indices for which they will determine nonzero values may be found by taking all of the possible products of the indices of nonzero ζ_i's with the indices of nonzero τ_j's. When the product of the indices of $\zeta_j\tau_k$ equals the product of the indices of $\zeta_l\tau_m$, we simply add the vertices that are to be placed in that index value by both products. Now to get the number of vertices to be placed in a given one of these index values we follow a similar procedure. Namely, *we multiply the values which correspond to the indices that were multiplied to get a given index, and place the product in that index. That is to say,* $\zeta_j \tau_k$ *will be placed at index* jk *in the wedge vector.*

Now, when we have pairs of vertices of the form (b_1, d_1) *and* (b_1, d_2) *we follow a slightly different operation to place them in the vector. To find the index, we take the degree of* b_1 *and multiply it by the value of* $\binom{x}{c}$. *To find how many vertices to place in this index, we use our previously determined formula for finding how many* d_i *share the same number of neighbors with* d_1 *as does* d_2*, which is* $\binom{d}{x}\binom{N-d}{d-x}$ *when* $x = |d_1 \cap d_2|$.

Similarly, when we have pairs of the form (b_1, d_1) *and* (b_2, d_1)*, we find the index with* $\binom{x}{a}$ *times* $\binom{d}{c}$. *We then know how many vertices to place in this index as follows* $\binom{b}{x}\binom{N-b}{b-x}$ *when* $x = |b_1 \cap b_2|$.

If the values from the $(b_1, d_1), (b_1, d_2)$ *and* $(b_1, d_1), (b_2, d_1)$ *cases fall in an index in the wedge vector which we already know to hold some vertices, then we simply add their values to the existing values in the given index.*

Method 2 *Given* $P_N(a, b)$ *and* $P_N(c, d)$*, with "white" vectors* $(\rho_0, \rho_1, \rho_2, \ldots, \rho_{\binom{N-b}{b-a}})$ *and* $(\eta_0, \eta_1, \eta_2, \ldots, \eta_{\binom{N-d}{d-c}})$ *respectively, then the white vector for* $P_N(a, b) \wedge P_N(c, d)$ *may be found through the following process:*

First, we know that we will have $\binom{N-a}{b-a}\binom{N-c}{d-c}$ *white vertices in the wedge product which much hold places in the wedge vector. In examining common neighbors, we first consider those pairs of vertices in the wedge which are of the form* (a_1, c_1) *and* (a_2, c_2) *where* $a_1 \neq a_2$ *and* $c_1 \neq c_2$*. For these vertices, the indices for which they will determine nonzero values may be found by taking all of the possible products of the indices of nonzero* ρ_i*'s with the indices of nonzero* η_j*'s. When the product of the indices of* $\rho_j \eta_k$ *equals the product of the indices of* $\rho_l \eta_m$*, we simply add the vertices that are to be placed in that index value by both products. Now to get the number of vertices to be placed in a given one of these index values we follow a similar procedure. Namely, we multiply the values which correspond to the indices that were multiplied to get a given index, and place the product in that index. That is to say,* $\rho_j \eta_k$ *will be placed at index* jk *in the wedge vector.*

Now, when we have pairs of vertices of the form (a_1, c_1) *and* (a_1, c_2) *we follow a slightly different operation to place them in the vector. To find the index, we take the degree of* a_1 *and multiply it by the value of* $\binom{N-y}{d-y}$, *where* $y = |c_1 \cup c_2|$. *To find how many vertices to place in this index, we use our previously determined formula for finding how many* c_i *share the same number of neighbors with* c_1 *as does* c_2, *which is* $\binom{c}{x}\binom{N-c}{c-x}$ *when* $x = |c_1 \cap c_2|$.

Similarly, when we have pairs of the form (a_1, c_1) *and* (a_2, c_1), *we find the index with* $\binom{N-c}{d-c}$ *times* $\binom{N-z}{b-z}$, *where* $z = |a_1 \cup a_2|$. *We then know how many vertices to place in this index as follows* $\binom{a}{x}\binom{N-a}{a-x}$ *when* $x = |a_1 \cap a_2|$.

If the values from the $(a_1, c_1), (a_1, c_2)$ *and* $(a_1, c_1), (a_2, c_1)$ *cases fall in an index in the wedge vector which we already know to hold some vertices, then we simply add their values to the existing values in the given index.*

For some p-graphs with $n = 7$, the Ivanov wedge vectors are listed in the appendix.

0.13 Building the Ivanov wedge vector: an example

$P_7(1,2) \wedge P_7(2,6)$

As noted in the appendix, for $P_7(2,6)$, the black vector is (0,0,0,0,0,0,0,0,0, 0,6,0,0,0,0,0), and the white vector is (0,0,0,10,10,0). Also, for $P_7(1,2)$, the black vector is (10,10,0) and the white vector is (0,6,0,0,0,0,0).

To calculate our black wedge vector, we first consider those pairs of vertices which do not have the same set occupying either both first coordinates or both second coordinates. Multiplication of the indices holding non-zero values in the two component graph vectors here gives us the indices which will hold non zero values in the wedge vector based on these vertices. Thus as we have non-zero values in the 10th index for the black vector of $P_7(2,6)$ and in the 0th and 1st indices for the black vector of $P_7(1,2)$, we will have non-zero values in at least the 0th and 10th places of the wedge vector. We find the values to go in these indices by similarly multiplying the values from the vectors of the factors. In this way we see that $6 \circ 10 = 60$ goes in 0th and the 10th place in the wedge vector.

We must now consider the other types of pairs of vertices which may arise in the wedge product. For those pairs of the form (b_1, d_1) and (b_1, d_2), we first consider the various possibilities for $x = |d_1 \cap d_2|$. With d equaling 6, x may equal 5. To find the index which will hold the vectors of this form we multiply the degree of the first coordinate, $\binom{2}{1} = 2$ by $\binom{5}{2} = 10$, which equals 20. At index 20 we will place $\binom{6}{5}\binom{1}{1} = 6$, as related to our previous formula stating how many d_i share the same number of neighbors with d_1 as does d_2 based on intersection. We follow a very similar operation for pairs of vertices of the form (b_1, d_1) and (b_2, d_1) to find they add 10 in the 0th place (to be added to the existing 60) and 10 in the 15th place.

Thus the final wedge vector for $P_7(1, 2) \wedge P_7(2, 6)$ will be (70,0,0,0, 0,0,0,0,0,0,60,0,0,0,0,10,0,0,0,0,6).

We proceed in a very similar fashion to find the white wedge vector.

0.14 Conclusions

From the work detailed here, we draw the following broad conclusions:

- Due to the set theoretic relations of containment present in p-graphs we may encode a great deal of information about them in simple combinatorial formulas involving binomial coefficients.

- Using such formulas we may determine the size and elements of the Ivanov vector, encoding information about common neighbors for both black and white vertices in a p-graph.

- Determinations we made about the Ivanov vector and common neighbors may be expanded and slightly redefined to apply to wedge products of p-graphs.

0.15 Further questions

We might consider taking the following routes in continuing our investigations of common neighbors and p-graphs:

- Consider questions similar to those addressed from the perspective of the common neighbor in the p-graph; that is, find convenient ways to discuss the number of vertices that have a given vertex as a common neighbor with another vertex.
- Consider the same questions as we have about p-graphs for a slight generalization, **t-graphs**, $T_n(a, b, c)$, which has vertices corresponding to sets of size a and b, with two sets connected when their intersection has size c.

Both of the above should be answerable through combinatorial arguments similar to the ones given.

0.16 Appendix: Selected Ivanov vectors for $n = 7$

$P_7(2, 6)$
black vector: (0,0,0,0,0,0,0,0,0,0,6,0,0,0,0,0)
white vector: (0,0,0,10,10,0)
$P_7(1, 2)$
black vector: (10,10,0)
white vector: (0,6,0,0,0,0)
$P_7(2, 6) \wedge P_7(1, 2)$
black vector: (70,0,0,0,0,0,0,0,0,0,60,0,0,0,0,10,0,0,0,0,6,0,0,0,0,0,0,0,0,0,0)
white vector: (0,0,0,60,60,60,0,0,0,0,0,0,0,0,0,0,0,0,10,0,0,0,0,0,0,0,0,0,0,0,0)

$P_7(3, 6)$
black vector: (0,0,0,0,0,0,0,0,0,0,6,0,0,0,0,0,0,0,0,0,0)
white vector: (0,4,18,12,0)
$P_7(1, 3)$
black vector: (4,18,12,0)
white vector: (0,0,0,0,0,6,0,0,0,0,0,0,0,0,0,0)

$P_7(3,6) \wedge P_7(1,3)$
black vector: (28,0,0,0,0,0,0,0,0,0,108, 0,0,0,0,0,0,0,0,0,90,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
white vector: (0,0,0,0,0,24,0,0,0,0,0,0,0,0,0,0,0,0,10,0,0,0,0,0,0,0,0,0,0,0,0)

$P_7(3,5)$
black vector: (0,10,0,0,10,0,0,0,0,0,0)
white vector: (4,18,12,0,0,0,0)
$P_7(2,3)$
black vector: (22,12,0,0)
white vector: (10,10,0,0,0)
$P_7(3,5) \wedge P_7(2,3)$
black vector: (462,132,10,120,0,0,0,0,0,0,0,0,10,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0)
white vector: (394,180,120,0,0,18,10,0,0,0,0,0,0,0,0,12,0,0,
0,0,0,0,0,0, 0,0,0,0,0,0,0)

As none of the black and white vectors for a given graph are the same, we know that each of these graphs is strictly bitransitive.