# Deterministic Methods for Detecting Redundant Linear Constraints in Semidefinite Programming

Daniel Stover

Department of Mathematics and Statistics

Northen Arizona University,Flagstaff, AZ 86001.

July 30, 2002

**Abstract**

It has been shown that the problem of deterministically classifying constraints in semidefinite programming (SDP) is NP complete. An SDP is constructed that makes it possible to extend some constraint classification results from linear programming (LP) to SDP. Necessary and sufficient conditions for constraint classification, and ultimately, a deterministic method for detecting such are applied to the problem.

# 1 Introduction

Optimization is a vital field of research today. It is a discipline that is rich in theoretical and applied issues. LP's have been the subject of extensive research since Dantzig first discovered the famous simplex method. SDP has received ever increasing amounts of attention since it has been shown to generalize LP and quadratically constrained quadratic programming (QCQP), and thus, encompasses a diverse range of theoretical and applied problems. Constraint classification is a scheme for categorizing constraints with respect to their importance to the solution set of the program. We will classify a constraint as *redundant* or *necessary* but not both. Formal definitions are forthcoming. Constraint classification is interesting because of the potential to reduce the cost of solving problems, the insight into a particular SDP classification provides, and the fulfillment of studying it for its own sake. Redundancy in LP is discussed briefly to elucidate points that will be important to our special problem.

**Redundancy in LP**

Consider the LP

$$\begin{aligned} min \quad & c^T x \qquad (1) \\ s.t. \quad & b_j + a_j^T x \geq 0 \ (j = 1, 2, \ldots, q), \\ & x \ unrestricted \end{aligned}$$

where $x$, $b_j$, and $c \in \Re^n$. Its properties in general and in regards to constraint classification are well known and well documented. A thorough survey of constraint classification in LP is given in ([8],[15]). Most importantly, the idea of feasible region, necessary constraint, and redundant constraint need to be qualitatively clear as they pertain to (1).

The feasible region of (1) is the set of all solutions $x$ that satisfy the constraints of (1).

A necessary constraint with respect to the feasible region of (1) is one whose removal would change the feasible region of (1). It is notable that such a removal could change the optimal solution of (1) depending on the nature of $c^T$.

A redundant constraint with respect to the feasible region of (1) is one whose removal would change the feasible region, and hence, the optimal solution to (1) would also not change.

**Semidefinite Programming**

Now, SDP is introduced, and then, constraint classification in SDP is discussed. Suppose that $A$ and $B$ are symmetric matrices over the field $\Re$. $A$ is called *positive definite* iff its eigenvalues are positive and *positive semidefinite* iff its eigenvalues are nonnegeative. The many useful properties of these classes of square matrices are well known and will be referenced throughout this work. The symbol $\succ$ $(\succeq)$ is the *Löwner partial order* for matrices. Thus, $A \succ (\succeq) B$ iff $A - B$ is *positive definite*(*positive semidefinite*). In particular, we write $A \succ (\succeq) 0$ iff $A$ is *positive definite*(*positive semidefinite*).

A *semidefinite program* is the optimization problem

$$\begin{aligned} min \quad & c^T x \\ s.t. \quad & A^{(j)}(x) = A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succeq 0 \ (j = 1, 2, \ldots, q) \end{aligned} \tag{2}$$

where $A^j(x) \in \Re^{m_j \times m_j}$ and $b, \ x \in \Re^n$. It has a dual problem of the form

$$\begin{aligned} max \quad & -\sum_{j=1}^{q} A_0^{(j)} \bullet Z_j \\ s.t. \quad & \sum_{j=1}^{q} A_i^{(j)} \bullet Z_j = c_i \ (i = 1, 2, \ldots, n), \\ & Z_j \succeq 0 \end{aligned} \tag{3}$$

where $Z_j$ is an $m_j \times m_j$ matrix of variables. $\Re^n$ is called the ambient space of the SDP. A single constraint is called a *linear matrix inequality* (LMI).

The above SDP and its dual are related by the optimality conditions:

$$A^{(j)}(x) \succeq 0; \tag{4}$$

$$Z_j \succeq 0,\ \sum_{i=1}^{m} A_i^{(j)} \bullet Z_j = c_i\ (i = 1, 2, \ldots, q);$$

$$A^{(j)}(x) \bullet Z_j = 0.$$

These conditions are collectively known as the "KKT conditions" and describe primal feasibility, dual feasibility, and complementary slackness. Under these conditions, a strong duality analogous to that in LP holds([2],[5]).

SDP generalizes LP, and indeed it is not difficult to write (1) as a program subject to LMI's. Informally, SDP extends LP by replacing the nonnegativity constraints of an LP problem with restrictions on the positive semidefiniteness of LMI constraints. As mentioned before, QCQP is also generalized by SDP ([5]). Because of its great degree of generality, many problems in diverse areas such as control theory, statistical and combinatorial optimization, and certain problems in mathematical physics in fields such as mechanics of materials all can be posed as SDP ([5]). Polynomial time interior point methods do exist to solve SDP efficiently([2],[4],[5]).

**Redundancy in SDP**

Again, it is important to have a qualitative understanding of the basic ideas behind constraint classification (i.e., feasible region, necessity, redundancy), and it may not come as any surprise at this point that these definitions are analogous to the LP case.

The feasible region of (2) is the set of all solutions $x$ that satisfy the constraints of (2). It is important to note that the feasible region is convex so that SDP is a convex optimization problem([5]).

A necessary constraint with respect to the feasible region of (2) is one whose removal would change the feasible region of (2). It is notable that such a removal could change the optimal solution of (2) depending on the nature of $c^T$.

A redundant constraint with respect to the feasible region of (2) is one whose removal would change the feasible region, and hence, the optimal solution to (2) would also not change.

The classification of a particular LMI as necessary or redundant is known as the *semidefinite redundancy problem* ([13]). Deterministically classifying all constraints of a particular SDP is governed by the following theorem([13]).

**Theorem 1.1** $A^k(x) \succeq 0$ $(k \in \{1, 2, \ldots, q\})$ *is redundant if and only if the semidefinite program*

$$\begin{aligned} min \quad & \lambda_{min}(A^{(k)}(x)) \\ s.t. \quad & A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succeq 0 \ (j = 1, 2, \ldots, q) \end{aligned} \tag{5}$$

*has an optimal solution* $x^*$ *satisfying* $\lambda_{min}(A^{(k)}(x^*)) \geq 0$.

Theorem 1.1 is NP hard to solve. The difficulty of solving (5) is a result of the constraints containing convex quadratic constraints (CQC) which are NP hard to classify with respect to other CQC([5]). Since (5) can contain these it is also NP hard to solve. Furthermore, it is known that $\lambda_{min}(A^{(k)}(x))$ is not convex([2],[9],[11]). Many SDP with nonconvex objective functions are NP complete.

In order to make classification efficient for large problems, Monte Carlo methods are used to probabilistically to detect the necessary constraints ([13]). These methods classify constraints until some stopping rule is satisfied ([13]). Although there is always some possibility of the sets mentioned above being incomplete, these methods can classify large numbers of constraints efficiently.

In this paper, a special SDP is defined that makes it possible to classify the linear constraints of an SDP in order to decrease the cost of classifying LMI in general and to extend some LP classification results([7]) to SDP. Finally, we find a deterministic method using theorems to be presented shortly to determine completely the set of necessary constraints of this special problem.

## 2 The Problem

The following is a special case of SDP that is used to isolate the linear constraints of an SDP, classify them, and extend certain results to the LMI of the SDP.

$$\begin{aligned} min \quad & c^T x \\ s.t. \quad & b_j + a_j^T x \geq 0 \ (j = 1, 2, \ldots, q), \\ & A(x) = A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0 \end{aligned} \tag{6}$$

where $A(x) \in \Re^{m_j \times m_j}$ and $b, \ x \in \Re^n$. It has the following dual formulation:

$$\begin{aligned} max \quad & -\sum_{j=1}^{q} b_j y_j - A_0 \bullet Z \\ s.t. \quad & \sum_{j=1}^{q} (a_j)_i y_j = c_i \ (i = 1, 2, \ldots, n), \end{aligned} \tag{7}$$

$$y_j \geq 0,$$

$$Z \succeq 0.$$

Now, LP results can be applied to the linear constraints of this problem and extended to the LMI.

The KKT conditions are:

$$\begin{aligned} b_j + a_j^T x \geq 0 \ (j = 1, 2, \ldots, q),\ A(x) \succeq 0; \\ \textstyle\sum_{j=1}^{q} (a_j)_i y_j = b_i \ (i = 1, 2, \ldots, n),\ y_j \geq 0,\ Z \succeq 0; \\ (b_j + a_j^T x) y_j = 0 \ (j = 1, 2, \ldots, q),\ A(x) \bullet Z = 0. \end{aligned} \tag{8}$$

Recalling some of the informal definitions described previously, formal definitions applied specifically to (6) are now given:

**Definition 2.1** *The feasible region $\mathcal{R}$ is the set of $n \times 1$ vectors that satisfy the constraints of (5). i.e., $\mathcal{R}= \{x \mid b_j + a_j^T x \geq 0,\ A(x) \succeq 0,\ j = 1, 2, \ldots, q\}$. Similarly, $\mathcal{R}_k$ is the set of vectors that satisfying the set of constraints of (5) but excluding the $k^{th}$ constraint so that $\mathcal{R}_k = \{x \mid b_j + a_j^T x \geq 0,\ A(x) \succeq 0,\ j = 1, 2, \ldots, k-1, k+1, \ldots, q\}$.*

**Definition 2.2** *A linear constraint of (5) $b_k - a_k^T x \geq 0$ $(k \in \{1, 2, \ldots, q\})$ is said to be redundant with respect to $\mathcal{R}$ iff $\mathcal{R}= \mathcal{R}_k$. Otherwise, it is said to be necessary. The same constraint, when redundant, is said to be weakly redundant if $\mathcal{R} \cap \mathcal{K} \neq \emptyset$ where $\mathcal{K}= \{x \mid b_k - a_k^T x = 0\}$. i.e., The $k^{th}$ constraint touches the feasible region but would not change it if removed. Otherwise, the $k^{th}$ constraint is strongly redundant.*

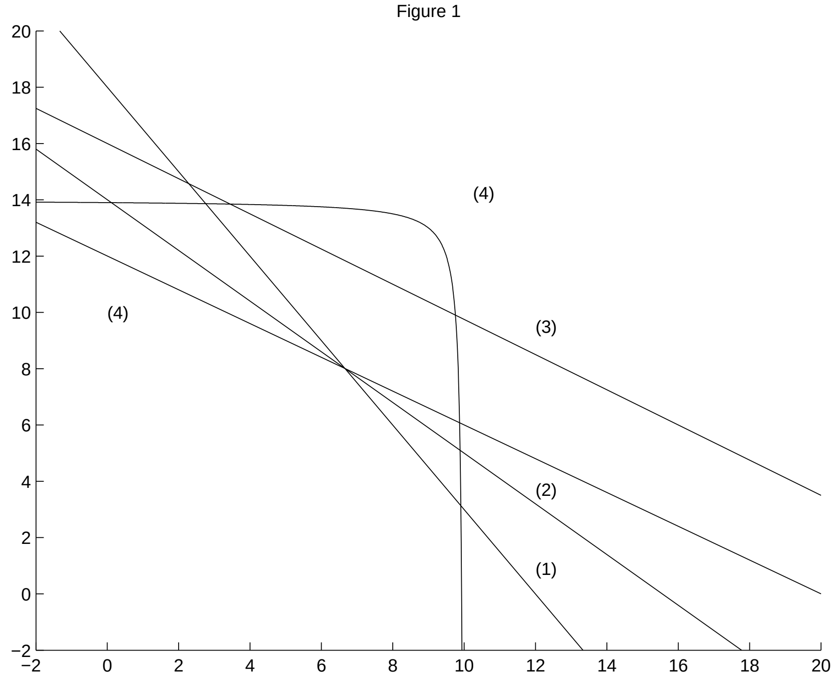
**Definition 2.3** *The LMI constraint of (5) $A(x) \succeq 0$ is said to be redundant with respect to $\mathcal{R}$ if $\mathcal{R}= \mathcal{R} \backslash A(x) \succeq 0$. Otherwise, it is said to be necessary. The same constraint, when redundant, is said to be weakly redundant if $\mathcal{R} \cap \mathcal{K} \neq \emptyset$ where $\mathcal{K}= \{x \mid A(x) \succeq 0, A(x) \not\succ 0\}$. i.e., The $A(x) \succeq 0$ touches the feasible region but would not change it if removed. Otherwise, the $A(x) \succeq 0$ is strongly redundant.*

Now that we have formally defined some ideas pertaining to (6) we will be prepared to begin discussing a method to deterministically classify the constraints of (6). Focus in this paper is on the classification of constraints and not on solving SDP. Thus, the objective function of (6) is of little concern here. Now, consider the feasible region

$$\begin{aligned} &36 - 3x_1 - 2x_2 \geq 0, \ (1) \\ &140 - 9x_1 - 10x_2 \geq 0, \ (2) \\ &128 - 5x_1 - 8x_2 \geq 0, \ (3) \\ &\begin{bmatrix} 10 & -1 & 0 \\ -1 & 14 & 0 \\ 0 & 0 & 60 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \succeq 0 \ (4) \end{aligned} \tag{9}$$

and its graph:

Figure 1




A method derived from the theorems presented shortly will be applied to this feasible region.

## 3 Methods

The following theorems reveal a method to classify constraints and improvements upon this method. First, we give necessary and sufficient conditions for redundancy, and in the process, set the foundation for our method.

**Theorem 3.1** *The linear constraint $b_k + a_k^T x \geq 0$ is redundant iff*

$$\begin{aligned} SDP_k: \quad & min \quad a_k^T x \\ & s.t. \quad b_j + a_j^T x \geq 0 \ (j = 1, 2, \ldots, k-1, k+1, \ldots, q), \\ & \qquad\quad A(x) \succeq 0 \end{aligned} \tag{10}$$

*has an optimal solution $x_k^*$ that satisfies $b_k + a_k^T x_k^* \geq 0$.*

**Proof** ($\Rightarrow$) Since the $k^{th}$ constraint is redundant, $\mathcal{R}_k = \mathcal{R}$. $SDP_k$ has a feasible region $\mathcal{R}_k$. The optimal solution $x_k^*$ of $SDP_k$ is on the boundary of $\mathcal{R}_k = \mathcal{R}$. Hence, $b_k + a_k^T x_k^* \geq 0$. ($\Leftarrow$) Suppose $b_k + a_k^T x \geq 0$ is necessary. Then, there exists a point $\tilde{x}$ such that $\tilde{x} \in \mathcal{R}_k$ but $b_k + a_k^T \tilde{x} < 0$. If $SDP_k$ has no optimal solution, then the proof is done. If $SDP_k$ has an optimal solution $x_k^*$, then $a_k^T x_k^* \leq a_k^T \tilde{x} \Rightarrow \ a_k^T x_k^* + b_k \leq a_k^T \tilde{x} + b_k < 0$. Thus, $a_k^T x_k^* + b_k \not\geq 0$.

**Remarks** Theorem 3.1 is the foundation of our method. If we apply Theorem 3.1 to each linear constraint and are supposing there are $q$ constraints, then we use this procedure $q$ times. The result would be the classification of all linear constraints but not the $LMI$ of our problem. Can this be improved? It can, and Theorem 3.2 and Conjecture 2 shows us how. First, definitions are now introduced that are necessary to the understanding and application of the following theorems. Notice that the definitions of "binding", "gradient", and "nondegenerate" are stated with respect to first the $LP$ case and then more generally to the $SDP$ case. It is important throughout this paper to recall that $SDP$ generalizes $LP$ and often has results analogous to $LP$. Although $SDP$ is more general, it is also more complicated. Thus, it can be helpful to have the simpler but analogous $LP$ conceptions to work with. Finally, "corner point" is defined followed by a definition of degeneracy for $SDP$ excluding LMI that have corner points. Corner points must be excluded for the time being in order to successfully extend $LP$ results to $SDP$. In application, these definitions are taylored to our problem (6).

**Definition 3.1** *A constraint* $b_k + a_k^T x \geq 0$ $(k \in \{1, 2, \ldots, q\})$ *of the* $LP$ (1) *is binding at* $x^*$ *if* $b_k + a_k^T x^* = 0$.

**Definition 3.2** *The gradient of a linear constraint* $b_k - a_k^T x \geq 0$ *is the* $n \times 1$ *vector*

$$\begin{aligned} \nabla a_k &= \begin{bmatrix} \frac{\partial a_1 x_1}{\partial x_1} & \ldots & \frac{\partial a_n x_n}{\partial x_n} \end{bmatrix}^T \\ &= \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}^T. \end{aligned}$$

**Definition 3.3** *([7]) An optimal solution* $x^*$ *to the* $LP$ (1) *is said to be* $Type$ 1 *nondegenerate if the gradients of the linear constraints binding on the solution are linearly independent.*

**Definition 3.4** *A constraint* $A^{(k)}(x) \succeq 0$ $(k \in \{1, 2, \ldots, q\})$ *of the* $SDP$ (3) *is binding on a solution* $x^*$ *if* $A^{(k)}(x^*) \succeq 0$ *and* $A^{(k)}(x^*) \not\succ 0$.

**Definition 3.5** *([14]) The* $i^{th}$ *component of the gradient of* $A^{(k)}(x^*) \succeq 0$ *is*

$$\begin{aligned} \nabla_i A^{(k)}(x^*) &= der \mid A^{(k)}(x^*) \mid \bullet A_i^{(k)} \\ &= tr((der \mid A^{(k)}(x^*) \mid)^T A_i^{(k)}) \end{aligned}$$

*where* $\mid A^{(k)}(x^*) \mid$ *is the determinant of* $A^{(k)}(x^*)$ *and* $\langle der \mid A^{(k)}(x^*) \mid \rangle_{ij} = \dfrac{\partial}{\partial a_{ij}^{(k)}(x^*)} \mid A^{(k)}(x^*) \mid$.

The definition of nondegeneracy in the SDP case is more involved than the LP case ([12]). Consider the Schur decompositon of the $j^{th}$ constraint $A^{(j)}(x) \succeq 0$ where $j = 1, 2, \ldots, q$ at an optimal solution $x^*$ so that

$$A^{(j)}(x^*) = Q_j diag(\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_{r_j}^k, 0, \ldots, 0) Q_j^T.$$

$A^{(j)}(x^*)$ necessarily has positive and zero eigenvalues since constraints evaluated at optimal points are strictly positive semidefinite. Thus,

$$rank(A^{(j)}(x^*)) = r_j.$$

Now, a matrix $\tilde{Q}_j$ is formed such that

$$\tilde{Q}_j = \textit{the submatrix given by the last } m_j - r_j \textit{ columns of } Q_j.$$

Next, let $J$ be the subset of constraints $A^{(j)}(x) \succeq 0$ that are binding on an optimal solution $x^*$ so that

$$\begin{aligned} J &= \{j \in \{1, 2, \ldots, q\} \mid A^{(j)}(x^*) \succeq 0,\ A^{(j)}(x^*) \not\succ 0\} \\ &= \{j_1, j_2, \ldots, j_v\}. \end{aligned}$$

Hence, the binding LMI's can be indexed as above. Now, let $L$ be the subset of linear constraints $b_l - a_l^T x \geq 0$ that are binding on an optimal solution $x^*$ so that

$$\begin{aligned} L &= \{l \in \{1, 2, \ldots, k-1, k+1, \ldots, q\} \mid b_l - a_l^T x^* = 0\} \\ &= \{l_1, l_2, \ldots, l_w\}. \end{aligned}$$

Hence, the binding linear constraints can be indexed as above. The nonzero variables of the dual SDP (3) at $x^*$ belong to a space of dimension $d$ such that

$$\begin{aligned} d &= [\sum_{j \in J} (m_j - r_j)(m_j - r_j + 1)/2] + [\sum_{l \in L} 1] \\ &= [\sum_{i=1}^{v} (m_j - r_j)(m_j - r_j + 1)/2] + w. \end{aligned}$$

From the above discussion and ([1], Theorem 3), we have the following definition.

**Definition 3.6** *([12]) An optimal solution $x^*$ of the SDP (3) is said to be Type* 2 *nondegenerate if*

$$diag(\tilde{Q}_{j_1}^T A_i^{(j_1)} \tilde{Q}_{j_1}^T, \ldots, \tilde{Q}_{j_v}^T A_i^{(j_v)} \tilde{Q}_{j_v}^T, (a_{l_1})_i, \ldots, (a_{l_w})_i)$$

*where $i = 1, 2, \ldots, n$ span a space of dimension $d$.*

It is notable that if degeneracy exists at some point, then we can associate its occurence to the presence of one or more weakly redundant constraints.

**Definition 3.7** *A constraint $A^{(k)}(x) \succeq 0$ $(k \in \{1, 2, \ldots, q\})$ is said to have a corner point at a boundary point $x^*$ of the feasible region of the SDP* (2) *if its gradient $\nabla A^{(j)}(x^*) = 0$.*

**Definition 3.8** *An optimal solution $x^*$ to the SDP* (3) *is said to be Type* 3 *nondegenerate if $A^{(j)}(x^*) \succeq 0$ is not a corner point and the gradients of the constraints binding on the solution are linearly independent.*

**Conjecture 1** A nondegenerate optimal solution $x^*$ of $SDP$ (3) that is not a corner point is $Type$ 3 *nondegenerate* if and only if it is $Type$ 2 *degenerate*.

**Remarks** We strongly believe this result holds. To truly bring completion to this method, it is very important to prove this conjecture so that our exclusion of corner points is justified. Of course, it would be even better to be able to handle corner points.

Now, we can finally introduce our first improvement.

**Theorem 3.2** *If the $k^{th}$ constraint of* (6) *is redundant and $x_k^*$ is a Type 3 nondegenerate optimal solution to the $SDP_k$* (10)*, then all constraints that are binding at $x_k^*$ are necessary.*

**Proof** Consider the set of all constraints binding at $x^*$. By $Definition$ 3.8 , the gradients of constraints binding at $x^*$ are linearly independent. By linear independence, if any of the binding constraints are removed, then the feasible region changes. Hence, the binding constraints are necessary.

**Remarks** First, let us note that the above proof is not rigorous. To be truly complete, analytic details need to be added. Secondly, it is our strong suspicion that this theorem holds for corner points as well, but its proof would entail the use of *Type* 2 *degeneracy*. Although the Type 2 definition is more general, it is also more difficult to use in application and analysis.

Now under the right conditions, Theorem 3.2 improves the method revealed by Theorem 3.1. When the $k^{th}$ constraint is redundant, the other constraints binding on the optimal solution $x_k^*$ of $SDP_k$ can be classified when there is no Type 3 degeneracy. In this way, we can classify the LMI. What if there is degeneracy at $x_k^*$? Many problems arise from degeneracy, and Conjecture 2 handles some of them.

**Conjecture 2** Let $I_k$ be the set of constraints binding at an optimal solution $x^*$ of (6). Suppose $\mid I_k \mid \leq n+1$, $t \in I_k$, and $t$ is a linear constraint. Consider the linear combination of gradients of constraints in $I_k$

$$\textstyle\sum_{i \in I_k \setminus t} u_i \nabla a_i = \nabla a_t \tag{11}$$

(i) If the solution to (11) is such that $u_i \geq 0$ for all $i \in I_k \setminus t$, then constraint $t$ is redundant. (ii) Furthermore, if $x_k^*$ is not a corner point of $A(x) \succeq 0$, then the other constraints binding on $x^*$, or all $i \in I_k \setminus t$, are necessary.

**Remarks** Conjecture 2 is able to classify the constraints binding on a degenerate optimal solution $x_k^*$ of $SDP_k$ whenever the weakly redundant constraint is not an LMI, there is no more than one weakly redundant constraint, and $x_k^*$ is not the corner point of an LMI. What are these corner points exactly? We have a definition, and as we apply our method to (9), an opportunity presents itself to discuss this further and see some visual examples of such.

Theorem 3.1 describes the process of solving $SDP_k$ where $k = 1, 2, \ldots, q$, and in the process, classifies completely the linear constraints of (6). Theorem 3.2 improves on this method by allowing the classification of the constraints binding on a solution of Theorem 3.1 under specified conditions. For example when solving $SDP_k$, we will find that the constraint $b_k + a_k^T x \geq 0$ is necessary or redundant. By Theorem 3.1 alone, we must solve $SDP_k$ $q-1$ more times. Whenever $b_k + a_k^T x \geq 0$ is necessary, we classify it as necessary and then solve $SDP_{k+1}$. If redundant, then we check for degeneracy at $x_k^*$ and form the set of binding constraints by checking feasibility at the optimal solution $x_k^*$ of $SDP_k$. If $x_k^*$ is nondegenerate, we may use Theorem 3.2 to classify the binding constraints as necessary. Our method is improved because we no longer need to solve $SDP_k$ with respect to these particular constraints. Also, we may be able to classify the LMI. If degeneracy exists, then one or more of the binding constraints are weakly redundant. Here, it gets messy. Conjecture 2 makes it possible to classify binding linear constraints on optimal solutions bound by at most $n+1$ constraints provided that the constraint causing the degeneracy is linear and $x^*$ is not a corner point. Again, if the redundant constraint at a degenerate point is a linear constraint, then we can classify it. If the redundant constraint at a degenerate point is an LMI, then this problem becomes NP hard to solve. Now, this process can be seen by applying it to (9). Applying our method to (9) will not only illustrate the mechanics of our method but will also give us a point of departure to illustrate through examples of degeneracy some of the difficulties generated by such.

$\mathbf{SDP_1}$ Let $\mathcal{R}_N$ be the set of necessary constraints and $\mathcal{R}_R$ be the set of redundant constraints. Then,

$$\min \begin{bmatrix} 3 & 2 \end{bmatrix}^T x \tag{12}$$

$$s.t. \quad 140 - 9x_1 - 10x_2 \geq 0, \ (2)$$

$$128 - 5x_1 - 8x_2 \geq 0, \ (3)$$

$$\begin{bmatrix} 10 & -1 & 0 \\ -1 & 14 & 0 \\ 0 & 0 & 60 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \succeq 0 \ (4)$$

By Sedumi, $SDP_1$ has an optimal solution at $x_1^* = \begin{bmatrix} 9.8876 & 5.1011 \end{bmatrix}^T$. Then, we evaluate

$$\begin{aligned} b_1 - (a_i)_1 x_1^* &= 36 - [3(9.8876) + 2(5.1011)] \\ &= -3.8650 \\ &\leq 0 \end{aligned}$$

Thus by Theorem 3.1, the $1^{st}$ constraint is classified as necessary so that $\mathcal{R}_N = \{1\}$ and $\mathcal{R}_R = \emptyset$. We proceed to $SDP_2$.

**$\mathbf{SDP_2}$**

$$\min \begin{bmatrix} 9 & 10 \end{bmatrix}^T x \tag{13}$$

$$s.t. \quad 36 - 3x_1 - 2x_2 \geq 0, \ (1)$$

$$128 - 5x_1 - 8x_2 \geq 0, \ (3)$$

$$\begin{bmatrix} 10 & -1 & 0 \\ -1 & 14 & 0 \\ 0 & 0 & 60 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \succeq 0 \ (4)$$

By Sedumi, $SDP_2$ has an optimal solution at $x_2^* = \begin{bmatrix} 20/3 & 8 \end{bmatrix}^T$. Then, we evaluate

$$b_2 - (a_i)_2 x_2^* = 140 - [9(20/3) + 10(8)]$$

$$= \quad 0$$

$$\leq \quad 0$$

Thus by Theorem 3.1, the $2^{nd}$ constraint can be added to $\mathcal{R}_R$. Since $SDP_2$ has found redundancy, we first find that $x_2^*$ is bound by the $1^{st}$ and $4^{th}$ constraints. Next, the definition of degeneracy is used to find that $x_2^*$ is nondegenerate. Thus by Theorem 3.2, the $1^{st}$ and $4^{th}$ constraints are classified as necessary so that $\mathcal{R}_N = \{1, 4\}$ and $\mathcal{R}_R = \{2\}$. We proceed to $SDP_3$.

Note that Theorem 3.2 is used as if our restriction regarding corner points does not exist. Again, it is our strong suspicion that this theorem holds for corner points as well, so I do not bother to check that $\nabla A(x_2^*) = 0$.

**SDP$_3$**

$$\begin{aligned} min \quad & \begin{bmatrix} 5 & 8 \end{bmatrix}^T x \\ s.t. \quad & 36 - 3x_1 - 2x_2 \geq 0, \ (1) \\ & 140 - 9x_1 - 10x_2 \geq 0, \ (2) \\ & \begin{bmatrix} 10 & -1 & 0 \\ -1 & 14 & 0 \\ 0 & 0 & 60 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \succeq 0 \ (4) \end{aligned} \tag{14}$$

By Sedumi, $SDP_3$ has an optimal solution at $x_3^* = \begin{bmatrix} 20/3 & 8 \end{bmatrix}^T x$

. Then, we evaluate

$$\begin{aligned} b_3 - (a_i)_3 x_3^* &= 128 - [5(20/3) + 8(8)] \\ &= 92/3 \\ &> 0 \end{aligned}$$

Thus by Theorem 3.1, the $3^{rd}$ constraint can be added to $\mathcal{R}_R$. Thus, $\mathcal{R}_N = \{1, 4\}$ and $\mathcal{R}_R = \{2, 3\}$, and having classified all constraints, the procedure ends. Let us continue as if we had not classified all constraints so that we can get a closer look at the less tidy possibilities of our procedure. Degeneracy and the complicated geometry of the LMI's are the source of these difficulties. Since $SDP_3$ has found redundancy, we first find that $x_3^*$ is bound by the $1^{st}$, $2^{nd}$, and $4^{th}$ constraints. Next, the definition of degeneracy is used to find that $x_3^*$ is degenerate. Thus, Theorem 3.2 is not applicable. Now, we compute the gradients of the constraints binding on $x_3^*$ and get

$$\begin{aligned} \nabla a_1 &= \begin{bmatrix} 3 & 2 \end{bmatrix}^T, \\ \nabla a_2 &= \begin{bmatrix} 9 & 10 \end{bmatrix}^T, \\ \nabla A(x_3^*) &= \begin{bmatrix} 57 & 95 \end{bmatrix}^T. \end{aligned}$$

In the process, we observe that $x_3^*$ is not a corner point since $\nabla A(x_3^*) \neq 0$. Since the weakly redundant constraint binding at $x_3^*$ is not an LMI, the number of binding constraints $\leq n + 1 = 2 + 1 = 3$, and $x_3^*$ is not a corner point, we can apply Conjecture 2. Since

$$1.6667 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 0.0702 \begin{bmatrix} 57 \\ 95 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix},$$

the $2^{nd}$ constraint is classified as redundant, and the $1^{st}$ and $4^{th}$ constraints are classified as necessary by Conjecture 2. This agrees with what we already know. In other similar problems, Conjecture 2 improves the method by classifying several constraints binding at one solution of $SDP_k$ by alleviating the need to solve $SDP^k$ for those particular constraints.

In short, the method can not improve on $SDP_k$ if the weakly redundant constraint of a degenerate point is an LMI or if the degenerate point is a corner point. The examples will make these difficulties more apparent.

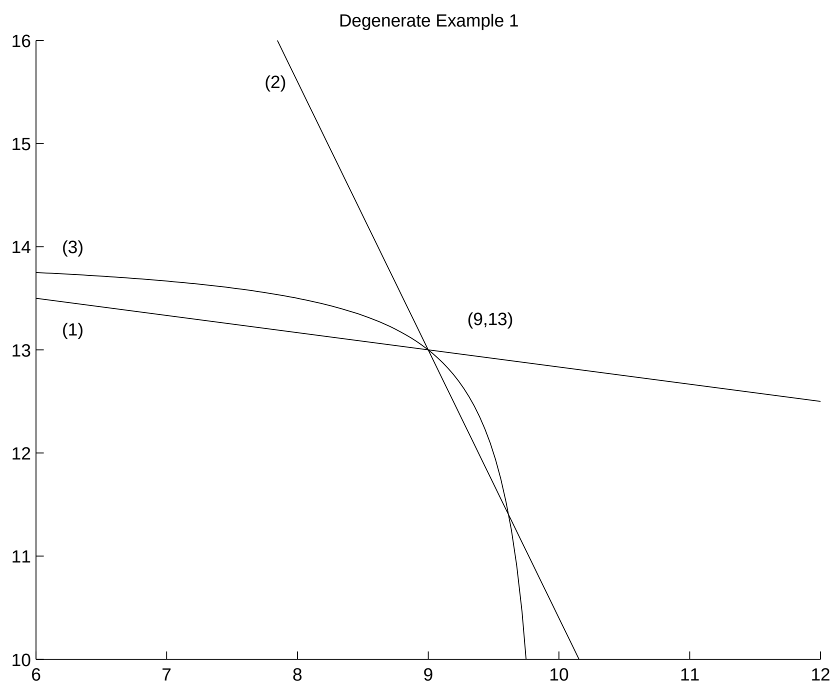
**Degenerate Example 1** Consider the feasible region

$$87/6 - 1/6x_1 - x_2 \geq 0, \ (1)$$

$$35 - x_1 - 2x_2 \geq 0, \ (2)$$

$$\begin{bmatrix} 10 & -1 \\ -1 & 14 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0 \ (3)$$

and its graph:



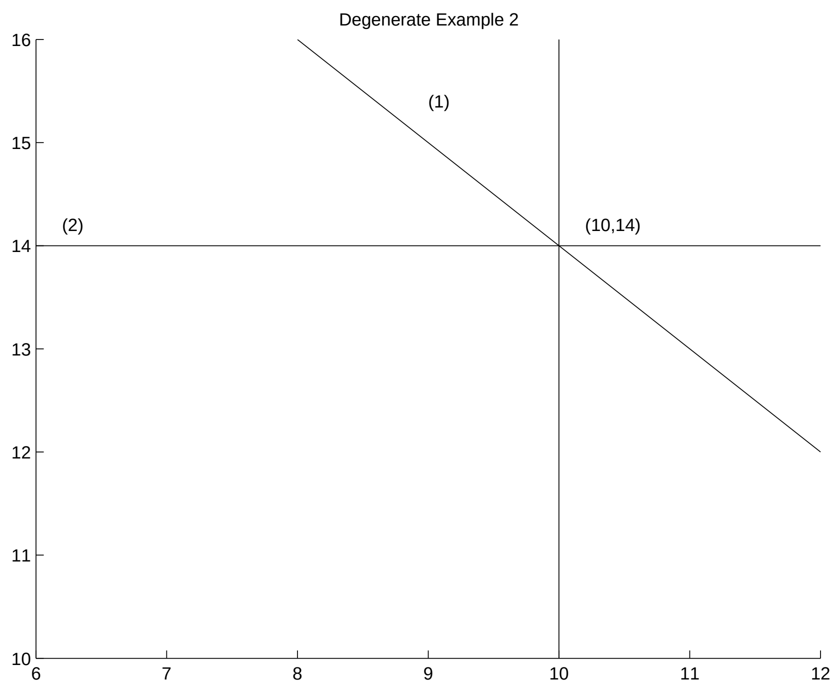

The $3^{rd}$ constraint causes the degeneracy at the boundary point (9, 13). If we allow Theorem 3.3 to be used to classify the constraints binding at (9, 13) facilitated by our definition of the gradient of an LMI, then it will classify the $3^{rd}$ constraint as redundant although it is not.

Now, the following is an example of degeneracy at a corner point.

**Degenerate Example 2** Consider the feasible region

$$24 - x_1 - x_2 \geq 0, \ (1)$$
$$\begin{bmatrix} 10 & 0 \\ 0 & 14 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0 \ (2)$$

and its graph:




The $1^{st}$ constraint causes degeneracy at the boundary point $(10,\ 14)$. If we allow Theorem 3.3 to be used to classify the constraints binding at $(10,\ 14)$ facilitated by our definition of the gradient of an LMI, then it can not classify the $1^{st}$ constraint at all because 11) will have no solution.

As the complexity of the geometry of LMI reveals itself in the examples of degeneracy seen above and through the construction of other degenerate examples, one begins to have an intuitive feel for the fact that deterministic LMI constraint classification is NP complete.

## 4 Conclusions

Deterministic methods for constraint classification in LP have been extended to $SDP$. By applying the method described in this work, one may classify all constraints of (6). Only in the cases that an optimal solution coincides with a corner point, that the number of constraints binding on an optimal solution exceeds the dimension of the ambient space plus one, and that the weakly redundant constraint at an optimal solution is an LMI can we not improve our method.

Future research in the method presented here could include techniques to successfully improve the method under the pathological cases described above especially concerning the occurence of corner points. Any such foray would have to seek a deeper understanding of the geometry of degeneracy in SDP. It is interesting that such a well mapped problem as the classifying of constraints in LP can get so complex simply with the addition of one LMI.

Also, research into other deterministic constraint classification methods could prove fruitful. One such method could use an extension of the *Turnover Lemma* to SDP as a foundational theorem and then improve there upon. Problems are already known in this approach ([13]), but it seems possible that an extension could be discovered.

It seems unlikely that one could sneak around the NP hardness of this problem, but maybe someone will (and get very famous).

## References

[1] F. Alizadeh, J. A. Haeberly, M. Overton, "Complementarity and Nondegeneracy in Semidefinite Programming", *Mathematical Programming*, Vol. 77, 1997, No. 2, Ser. B, pp. 111-128.

[2] F. Alizadeh, "Interior Point Methods in Semidefinite Programming with Applications to Combinatorial Optimization", *SIAM Journal on Optimization*, Vol. 5, No. 1, 1995, pp. 13-51.

[3] S. Boyd, L. El Ghaoui, "Linear Matrix Inequalities in System and Control Theory", *SIAM*, Vol. 15, Philadelphia, PA, 1994.

[4] S. Boyd, L. Vandenberghe, "Primal-Dual Potential Reduction Method for Problems Involving Matrix Inequalities", *Mathematical Programming*, Vol. 69, 1995, pp. 205-236.

[5] S. Boyd, L. Vandenberghe, "Semidefinite Programming", *SIAM Review*, vol. 31, no. 1, March 1996, pp. 49-95.

[6] S. Boyd, L. Vandenberghe, "Semidefinite Programming Relaxations of Non-Convex Problems in Control and Combinatorial Optimization", *Communications, Computation, Control and Signal Processing: A Tribute to Thomas Kailath (A. Paulraj et al., Ed.)*, 1997.

[7] R. J. Caron, "A Degenerate Extreme Point Strategy for the Classification of Linear Constraints as Redundant or Necessary", *Journal of Optimization Theory and Applications*, vol. 62, 1989, pp. 225-237.

[8] R. J. Caron, A. Boneh, S. Boneh, "Redundancy", *Advances in Sensitivity Analysis and Parametric Programming*, Kluwer Academic Publishers, International Series in Operations Research and Management Science, Vol. 6, 1997.

[9] J. Cullum, W. E. Donath, P. Wolfe, "The Minimization of Certain Nondifferentiable Sums of Eigenvalue Problems", *Mathematical Programming*, Vol. 3, 1975, pp.35-55.

[10] G. Gruber, S. Kruk, F. Rendl, H. Wolkowicz,"Presolving for Semidefinite Programs Without Constraint Qualifications", Technical Report CORR 98-32, University of Waterloo, Waterloo, Ontario, 1998.

[11] R. A. Horn, C. R. Johnson, "*Matrix Analysis*" Cambridge University Press, Cambridge, New York, Melbourne, 1985.

[12] S. Jibrin, "Redundancy in Semidefinite Programming", Carleton University, Canada, 1998.

[13] S. Jibrin, I. S. Pressman, "Monte Carlo Algorithms for the Detection of Necessary Linear Matrix Inequality Constraints", *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 2, 2001, pp. 139-154.

[14] S. Jibrin, J. Swift, "The Boundary of Weighted Analytic Centers for Linear Matrix Inequalities", Working Paper, Northern Arizona University, 2002.

[15] M. H. Karwan, V. Lotfi, J. Telgen, S. Zionts, "*Redundancy in Mathematical Programming: A State-of-the-Art Survey*", Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983

[16] K. G. Murty, S. N. Kabadi, and R. Chandrasekaran, "Infeasibility Analysis for Linear Systems, a survey", *Invited Paper for the Arabian Journal of Science and Technology*, Special Issue on Optimization, 2000.