General Results Concerning a Family of Worthy Semi-Symmetric Graphs*

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Abstract

The topic of semi-symmetric graphs is an area of current research in algebraic graph theory. In this paper, we will discuss a general construction of semi-symmetric graphs, using the bi-transitive disjunction graphs $D_N(a, b)$, and then show what very minimal conditions imposed on pairs of these graphs will give us semi-symmetric products.

1 Background and Preliminaries

1.1 Definitions

In this paper, we will consider simple and connected graphs. Every graph that we discuss will be *bipartite*. We will also assume that each graph is already bi-colored, having black and white vertices, where each edge has one black vertex and one white vertex as endpoints.

An *automorphism* or *symmetry* of a graph Γ is a permutation of its vertices which preserves edges. $Aut(\Gamma)$, the collection of all automorphisms of Γ, is a group under composition. One thing to note is that $Aut(\Gamma)$ acts on the vertices as well as the edges of Γ. In a bipartite graph, define $Aut^+(\Gamma) \leq Aut(\Gamma)$ to be the subgroup consisting of all the color-preserving symmetries. A bipartite graph is then called *bi-transitive* provided that $Aut^+(\Gamma)$ acts transitively on the edges of Γ but not the vertices. In a bi-transitive graph, $Aut^+(\Gamma)$ will also act transitively on the vertices of each color. If, however, $Aut^+(\Gamma) = Aut(\Gamma)$, we say that Γ is *strictly* bi-transitive.

Let Γ be bi-transitive, and suppose that Γ has B black vertices and W white vertices. Since all the black vertices are in the same orbit, it follows that they all have the same degree, k, and similarly, each white vertex must also have the same degree, e. Note that the number of edges in Γ is given by the expression

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Bk, since each black vertex has k edges. Likewise, the expression We also counts the edges in Γ, and hence, $Bk = We$. Notice that if $k \neq e$ (and hence, $B \neq W$), our graph Γ is strictly bi-transitive. However, if $k = e$ (and so $B = W$), the graph is *regular*; each vertex has the same degree. If Γ is regular and strictly bi-transitive, we say that Γ is *semi-symmetric*.

Finally, we say that a graph is *worthy* provided that no two vertices have exactly the same neighbors. Graphs that are unworthy can pose a problem in the study of the symmetries of graphs. If there is one set of vertices that share common edges with another set of vertices, one can interchange vertices among each set, thus preserving the structure of the graph. This allows the graph to have symmetries of a local nature that have nothing to do with the global structure of the graph. In this paper, we wish to study graphs of a worthy nature.

1.2 Products of Graphs

Let Γ_1 and Γ_2 be two graphs. There are many ways of constructing the product of two graphs; one such product is called the *categorical product*, among other names. In this construction, the graph $\Gamma_1 \times \Gamma_2$ has vertex set $V(\Gamma_1) \times V(\Gamma_2)$, and two vertices (a, b) and (c, d) are adjacent provided that $\{a, c\}$ is an edge of Γ_1 and $\{b, d\}$ is an edge of Γ_2. If both graphs are bi-colored, $\Gamma_1 \times \Gamma_2$ is not connected, and so we must modify the definition. Define $\Gamma_1 \wedge \Gamma_2$ t be the connected component of $\Gamma_1 \times \Gamma_2$ containing all vertices (a, c) where a and c are either both black or both white. We call the graph $\Gamma_1 \wedge \Gamma_2$ the *wedge product* of Γ_1 and Γ_2.

There are a couple of properties of the wedge product that we must note. First, the following result:

Theorem 1. *If Γ_1 and Γ_2 are both bi-transitive, where Γ_1 has B_1 black vertices of degree k_1 and W_1 white vertices of degree e_1, and Γ_2 has B_2 black vertices of degree k_2 and W_2 white vertices of degree e_2, their wedge product is also bi-transitive, with B_1B_2 black vertices of degree k_1k_2 and W_1W_2 white vertices each of degree e_1e_2.*

Another property to note of the wedge product of two graphs Γ_1 and Γ_2 is that the two graphs $\Gamma_1 \wedge \Gamma_2$ and $\Gamma_2 \wedge \Gamma_1$ are isomorphic. Finally, it is noteworthy to mention that the wedge product of two worthy graphs is also worthy.

1.3 The Graphs $D_N(a, b)$

The rest of this paper will center around a particular type of bipartite graph, $D_N(a, b)$, and the resulting wedge products. If a, b, and N are positive integers satisfying $a + b < N$, and $[N]$ is the set $\{1, 2, \ldots, N\}$, we define the graph $D_N(a, b)$ to be the bipartite graph where the black vertices are subsets of $[N]$ of size b and the white vertices are subsets of $[N]$ of size a. A set A of size a (a white vertex) is connected to a set B of size b (a black vertex) when the sets A and B are disjoint. An example of such a graph is shown in Figure 1.

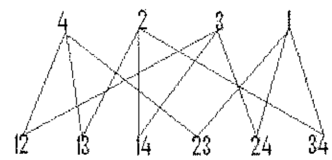

Figure 1: $D_4(1,2)$

It is clear that in the graph $D_N(a,b)$, the number of black vertices, B, and the number of white vertices, W, is

$$B = \binom{N}{b} \qquad \text{and} \qquad W = \binom{N}{a}.$$

Consider a black vertex. Since this vertex has b different numbers associated with it, remove them from $[N]$. The remaining $N - b$ numbers are those that the black vertex can be associated (share an edge) with. Then, the number of white vertices that the black vertex is connected to is the number of subsets of the remaining $N - b$ numbers of size a. The number of those white vertices is the degree, k, of a black vertex. Do the same for a white vertex to determine its degree, e. It therefore follows that

$$k = \binom{N-b}{a} \qquad \text{and} \qquad e = \binom{N-a}{b}.$$

Note that the graph $D_N(a,b)$ is bi-transitive. Another thing to note here is that if $a \neq b$, then $B \neq W$, and the graph is strictly bi-transitive but not regular. However, if $a = b$, then $B = W$, and switching labels on the black and white vertices is a symmetry, so our graph is regular but not strictly bi-transitive. Also, we see that $D_N(a,b)$ is worthy and connected for all a, b, N with $a + b < N$.

1.4 Regular Wedge Products of $D_N(a,b)$

From Theorem 1, we see that if we want to use the D_N's to construct a regular, bi-transitive graph, we must satisfy the condition $k_1 k_2 = e_1 e_2$ (hence, $B_1 B_2 = W_1 W_2$).

Consider two graphs, $D_N(a,b)$ and $D_M(c,d)$, with $a < b < N/2$. In $D_N(a,b)$, we have

$$B_1 = \binom{N}{b}, \qquad k_1 = \binom{N-b}{a}, \qquad W_1 = \binom{N}{a}, \qquad e_1 = \binom{N-a}{b}.$$

In $D_M(c,d)$, we have

$$B_2 = \binom{M}{d}, \qquad k_2 = \binom{M-d}{c}, \qquad W_2 = \binom{M}{c}, \qquad e_2 = \binom{M-c}{d}.$$

In the wedge product, $D_N(a,b) \wedge D_M(c,d)$, it follows that

$$B = \binom{N}{b}\binom{M}{d} \quad \text{and} \quad W = \binom{N}{a}\binom{M}{c}$$

and

$$k = \binom{N-b}{a}\binom{M-d}{c} \quad \text{and} \quad e = \binom{N-a}{b}\binom{M-c}{d}.$$

Therefore, if we want the wedge product to be regular and bi-transitive, the following conditions must be satisfied:

$$\binom{N}{b}\binom{M}{d} = \binom{N}{a}\binom{M}{c} \tag{1}$$

and

$$\binom{N-b}{a}\binom{M-d}{c} = \binom{N-a}{b}\binom{M-c}{d}. \tag{2}$$

Notice, however, that condition 1 is equivalent to condition 2, so we will use the first form, 1, to refer to the regularity of the wedge product. If we take condition 1 and rewrite it as

$$\frac{\binom{N}{b}}{\binom{N}{a}} = \frac{\binom{M}{c}}{\binom{M}{d}},$$

we can draw some conclusions on both c and d. On the left of the equation, we see that as $a < b < N/2$, this implies that the numerator is larger than the denominator. This must also be the case on the right-hand side of the equation – this means that we also assume $d < c < M/2$.

Now we can ask the question, that given condition 1, when is the graph $D_N(a,b) \wedge D_M(c,d)$ semi-symmetric? That is, we can rephrase the question and ask when the wedge product is strictly bi-transitive. Since $D_N(a,b) \wedge D_M(c,d)$ is regular (by condition 1) and bi-transitive (by Theorem 1), determining strict bi-transitivity is enough. In order to answer this question, there is a tool that we must first discuss.

1.5 The Ivanov Vectors

In [1], Ivanov developed a pair of vectors useful for encoding information about common neighbors of vertices. For a bi-transitive colored graph Γ, he introduced two a vectors: the ab-vector for the blacks, and the aw-vector for the whites. We modify and expand his definition here to suit our needs.

The vector ab has length $k+1$, with entries index 0 through k, and the vector aw has length $e+1$, indexed 0 through e. These vectors are calculated on a base vertex of their color. In the i^{th} entry of the ab-vector is the number of black vertices which share i common neighbors with the base vertex. The aw-vector is similarly defined. However, since Γ is bi-transitive, notice that the vector will not depend on the choice of the base vertex.

Also, we wish to point out that the sum of the entries in the ab-vector is B, and in the aw-vector, W. What is important to note about these vectors is that the graph Γ is strictly bi-transitive if the black and white vectors are not the same. Studying these vectors will give us information on how to determine when the product $D_N(a,b) \wedge D_M(c,d)$ is strictly bi-transitive.

Consider again the graph $D_4(1,2)$. If we choose the base vertex $\{12\}$, we can construct the ab vector. The entry in position 0 of the vector is the number of black vertices (including $\{12\}$) that share exactly 0 common neighbors with $\{12\}$. Notice that the only other vertex is $\{34\}$. Therefore, we place an entry of 1 at index 0 of the vector. Also, the number of black vertices that share exactly 1 common neighbor with $\{12\}$ is 4 (they are $\{13\}$, $\{14\}$, $\{23\}$, and $\{24\}$). Finally, the only vertex that shares exactly 2 common neighbors with $\{12\}$ is itself, so the vector gets an entry of 1 at index 2. Therefore, we see that the black vector is $ab = (1,4,1)$. If we do the same for the white vector, we see that it is $aw = (0,3,0,1)$.

The larger a graph gets, the more zeros each vector will have, so we introduce a different notation, due to Wilson [3]: let $\{i,c\}$ stand for the vector of appropriate length, whose i^{th} entry is c and with zeros at every other index. With this new notation, we see that $ab = \{0,1\} + \{1,4\} + \{2,1\}$ and $aw = \{1,3\} + \{3,1\}$.

In order to give a generic description of the Ivanov vectors for $D_N(a,b)$, we first need a couple of results, both due to Wilson [3]. I cite the following without proof:

Lemma 1. *For $0 \leq s \leq b$, the number of sets B_2 which intersect a given set B_1, both of size b, in a set of size s, is*

$$\binom{b}{s}\binom{N-b}{b-s}.$$

The number of common neighbors of a pair of such black vertices is

$$\binom{N-2b+s}{a}.$$

We see that a similar result holds for the white vertices. With this result, we can now construct our vector as follows:

Theorem 2. *In the graph $D_N(a,b)$,*

$$ab = \sum_{s=0}^{b} \left\{ \binom{N-2b+s}{a}, \binom{b}{s}\binom{N-b}{b-s} \right\}$$

and

$$aw = \sum_{s=0}^{a} \left\{ \binom{N-2a+s}{b}, \binom{a}{s}\binom{N-a}{a-s} \right\}.$$

Finally, if we have two bi-transitive graphs Γ_1 and Γ_2, we can construct the Ivanov vectors of their wedge product as follows:

Theorem 3. *If Γ_1 has the black vector given as*

$$a1b = \sum_i \{i, a1b_i\}$$

and Γ_2 has the black vector given as

$$a2b = \sum_j \{j, a2b_j,$$

then their wedge product $\Gamma_1 \wedge \Gamma_2$ has the black vector

$$ab = \sum_i \sum_j \{ij, a1b_i a2b_j\}.$$

A similar result will hold for the white vector as well.

We define the *leading index* of an Ivanov vector to be the index of the first non-zero entry of that vector. As an example, consider $D_4(1,2)$. The ab-vector is given as $(1,4,1)$ and the aw-vector is $(0,3,0,1)$. Then, the leading index of the ab-vector is 0, while the leading index of the aw-vector is 1.

We also define the *critical index* of an Ivanov vector to be the index of the second-to-last non-zero entry of that vector. The critical index of the ab vector of $D_4(1,2)$ is 1, as well as for aw.

There is one thing to notice about the Ivanov vectors. If the vectors for a bi-transitive graph are different, this implies that the graph is strictly bi-transitive (one of the conditions for semi-symmetry). However, the converse is not the case, nor is it the case that all semi-symmetric graphs have unequal Ivanov vectors.

2 Regular Products of D_N's

If we want to find semi-symmetric graphs, we first need to make sure that the graphs are regular. In the wedge product $D_N(a,b) \wedge D_M(c,d)$, in order for the product to be regular, we must satisfy condition 1. Now, consider the following:

Lemma 2. *If r and k are both integers, with $r \geq 1$ and $k \geq 0$, then*

$$\frac{\binom{r+(r+1)k}{k+1}}{\binom{r+(r+1)k}{k}} = r.$$

Proof. Using the factorial form of the binomial coefficient, we see

$$\begin{aligned}
\frac{\binom{r+(r+1)k}{k+1}}{\binom{r+(r+1)k}{k}} &= \frac{\frac{(r+rk+k)!}{(k+1)!(r+rk-1)!}}{\frac{(r+rk+k)!}{k!(r+rk)!}} \\
&= \frac{k!(r+rk)!}{(k+1)!(r+rk-1)!} \\
&= \frac{r+rk}{k+1} \\
&= \frac{r(k+1)}{k+1} \\
&= r.
\end{aligned}$$

□

Thus, in the $r+(r+1)k$th row of Pascal's Triangle, there are two consecutive entries whose ratio is r. So, in row $r+(r+1)l$ (where $l \neq k$), choose another two consecutive entries whose ratio is r. If we choose these rows and entries according to the above lemma, we can find wedge products of D_N's which are regular and satisfy

$$\binom{N}{b}\binom{M}{d} = \binom{N}{a}\binom{M}{c}.$$

Throughout this section, we consider r, k, and l. Let $r \geq 2$ and $k \geq 1$. Let

$$N = r + (r+1)k,$$

$$a = k,$$

and

$$b = k + 1.$$

Notice that by Lemma 2, we have

$$\frac{\binom{N}{b}}{\binom{N}{a}} = r. \tag{3}$$

Let $l > k$. If we now define

$$M = r + (r+1)l$$

and let

$$c = l + 1$$

and

$$d = l,$$

then by Lemma 2 we have

$$\frac{\binom{M}{c}}{\binom{M}{d}} = r. \tag{4}$$

Notice that if $l = k$, the resulting graph will be not be strictly bi-transitive, and therefore not semi-symmetric. For $l < k$, similar results follow the case when $l > k$.

Setting equations 3 and 4 equal and rewriting, we get

$$\binom{N}{b}\binom{M}{d} = \binom{N}{a}\binom{M}{c}, \tag{5}$$

and hence, the graph of $D_N(a,b) \wedge D_M(c,d)$ satisfies condition 1 and is regular. Notice, however, with $a + b < N$ and $c + d < M$, and $a < b < N/2$ and $d < c < M/2$, there are three other ways to construct regular wedge products:

$$\binom{N}{N-b}\binom{M}{d} = \binom{N}{a}\binom{M}{c}, \tag{6}$$

$$\binom{N}{b}\binom{M}{d} = \binom{N}{a}\binom{M}{M-c}, \text{ and} \tag{7}$$

$$\binom{N}{N-b}\binom{M}{d} = \binom{N}{a}\binom{M}{M-c}. \tag{8}$$

These other constructions will be discussed in later sections.

Finally, before we start analyzing the different constructions, define the graph

$$D(a,b,c,d;N,M)$$

to be $D_N(a,b) \wedge D_M(c,d)$ with a,b,c,d,N, and M defined as above.

We now will show properties of the Ivanov vectors for this graph $D(a,b,c,d;N,M)$.

2.1 First Construction: $D(a,b,c,d;N,M)$

For this construction, we will need the following information. Notice that

$$a + rb = N \tag{9}$$

and also,

$$rc + d = M. \tag{10}$$

Theorem 4. *Given the construction $D(a,b,c,d;N,M)$, the leading index of the ab and aw vectors are never 0.*

Proof. Assume that the leading index of the ab-vector for the wedge product is 0. Then it must be the case that either the leading index of the ab-vector of $D_N(a,b)$ is 0, that the leading index of the ab-vector of $D_M(c,d)$ is 0, or both are 0. We will show that this cannot be the case. Consider $D_N(a,b)$, and assume that the leading index is zero. Then, by Theorem 2, we see that this index is $\binom{N-2b}{a}$ (this occurs when $s = 0$). For this term to be zero, it must be that $N - 2b < a$. However, in Equation (9), we see that

$$\begin{aligned} N &= a + rb \\ &\geq a + 2b \\ N - 2b &\geq a. \end{aligned}$$

Therefore, it can never be the case that $N - 2b < a$. Hence, the leading index of the ab vector of $D_N(a, b)$ can never be 0.

Now, consider $D_M(c, d)$, and also assume that the leading index of the ab-vector is 0. This index is also given by Theorem 2 as $\binom{M-2d}{c}$ (also when $s = 0$). For this term to be zero, it must be the case that $M - 2d < c$. We will also show that this can never be the case. Consider Equation (10):

$$\begin{aligned} M &= rc + d \\ &\geq 2c + d \\ &> c + 2d \\ M - 2d &\geq c. \end{aligned}$$

It therefore follows that the leading index of the ab-vector of $D_M(c, d)$ can never be 0. By Theorem 3, we see that the leading index of the ab-vector of $D(a, b, c, d; N, M)$ is the product of the two leading indices of the ab-vectors of$D_N(a, b)$ and $D_M(c, d)$. Since these indices are never 0, we see that their product can never be 0. Therefore, the leading index of the ab-vector of $D(a, b, c, d; N, M)$ is never 0.

Similar reasoning will show the same holds for the aw-vector of the wedge product. □

Theorem 5. *For all $r \geq 2$, $D(a, b, c, d; N, M)$ has leading index greater in the ab-vector than in the aw-vector, showing semi-symmetry.*

Proof. The leading index of the ab-vector of $D(a, b, c, d; N, M)$ is given as

$$\binom{N-2b}{a}\binom{M-2d}{c} \tag{11}$$

while the leading index of the aw-vector is

$$\binom{N-2a}{b}\binom{M-2c}{d}. \tag{12}$$

Substituting in relationships for N, M, a, b, c, and d, and simplifying, Equations 11 and 12 become, respectively,

$$\binom{(k+1)(r-1)-1}{k}\binom{(l+1)(r-1)+1}{l+1}$$

and

$$\binom{(k+1)(r-1)+1}{k+1}\binom{(l+1)(r-1)-1}{l}.$$

Let $A = (k+1)(r-1)$ and $B = (l+1)(r-1)$. Using the definition for the binomial coefficient and simplifying, the leading index of the ab-vector becomes

$$\frac{(A-1)!(B+1)!}{(k)!(l+1)!(A-1-k)!(B-l)!}, \tag{13}$$

while the leading index of the aw-vector becomes

$$\frac{(A+1)!(B-1)!}{(k+1)!(l)!(A-k)!(B-1-l)!}. \tag{14}$$

Now we claim that the leading index for the ab-vector is greater than the leading index of the aw-vector. We will show this by showing their ratio is greater than one. Let R be the ratio of expression 13 to expression 14. Simplifying, we get

$$\begin{aligned} R &= \frac{\frac{(A-1)!(B+1)!}{(k)!(l+1)!(A-1-k)!(B-l)!}}{\frac{(A+1)!(B-1)!}{(k+1)!(l)!(A-k)!(B-1-l)!}} \\ &= \frac{(l)!(k+1)!}{k!(l+1)!} \cdot \frac{(A-1)!(B+1)!}{(A+1)!(B-1)!} \cdot \frac{(A-k)!(B-l-1)!}{(A-k-1)!(B-l)!} \\ &= \frac{k+1}{l+1} \cdot \frac{(B+1)B}{(A+1)A} \cdot \frac{A-k}{B-l}. \end{aligned} \tag{15}$$

Since $A = (k+1)(r-1)$, divide by $(r-1)$ to solve for $(k+1)$ and substitute into (15). Do likewise to solve for $(l+1)$. Then R becomes

$$\begin{aligned} R &= \frac{\frac{A}{r-1}}{\frac{B}{r-1}} \cdot \frac{(B+1)B}{(A+1)A} \cdot \frac{A-k}{B-l} \\ &= \frac{A}{B} \cdot \frac{(B+1)B}{(A+1)A} \cdot \frac{A-k}{B-l} \\ &= \frac{(B+1)(A-k)}{(A+1)(B-l)}. \end{aligned} \tag{16}$$

Recall $A = (k+1)(r-1)$. Solve for k to obtain

$$k = \frac{A}{r-1} - 1. \tag{17}$$

Also, solve for l in terms of B and r to obtain

$$l = \frac{B}{r-1} - 1. \tag{18}$$

Substitute (17) and (18) into Equation 16 to obtain

$$\begin{aligned} R &= \frac{(B+1)(A-k)}{(A+1)(B-l)} \\ &= \frac{(B+1)(A-\frac{A}{r-1}+1)}{(A+1)(B-\frac{B}{r-1}+1)} \\ &= \frac{AB+A+B+1-\left(\frac{AB+A}{r-1}\right)}{AB+A+B+1-\left(\frac{AB+B}{r-1}\right)}. \end{aligned} \tag{19}$$

Recall that since it was assumed that $l > k$, it follows that $B > A$. It then follows that $\frac{AB+A}{r-1} < \frac{AB+B}{r-1}$, and so the right-hand side of (19) is greater than 1. Therefore, the leading index of the ab-vector is greater than the leading index of the aw-vector. This shows that the two vectors are not equal, leading us to conclude that for all $r \geq 2$, the graph $D(a, b, c, d; N, M)$ is semi-symmetric. □

Notice also that if instead we restricted the construction of $D(a, b, c, d; N, M)$ to the case when $k > l$, this same result would follow, however in this case, we would see that the leading index of the aw-vector is larger than the leading index of the ab-vector.

2.2 Second Construction: $D(a, N-b, c, d; N, M)$

If we construct our next wedge product in the same way as $D(a, b, c, d; N, M)$, but this time, replace b with $N - b$, we satisfy the second construction for regularity given in Equation (6). It therefore follows that the graph $D(a, N - b, c, d; N, M)$ is regular and bi-transitive. Also, we see that the following relationships hold for $D(a, N - b, c, d; N, M)$:

$$a + b + 1 = N$$

and

$$rc + d = M.$$

We will now show that the two Ivanov vectors are unequal.

Theorem 6. *The leading index of the ab-vector is greater than that of the aw-vector in the graph $D(a, N - b, c, d; N, M)$.*

Proof. The leading index of the ab-vector will occur with the product of the leading index of the ab-vector of the graph $D_N(a, N - b)$ and the leading index of the ab-vector of the graph $D_M(c, d)$. Likewise for the aw-vector. We begin our search of where these occur.

Clearly the leading index for the ab-vector of $D_M(c, d)$ occurs at $\binom{M-2d}{c}$ and likewise, the leading index for the aw-vector of $D_M(c, d)$ occurs at $\binom{M-2c}{d}$ as was shown in the proof of Theorem 4.

It now remains to show the case for both ab- and aw-vectors for $D_N(a, N-b)$. Recall that the indices for the ab-vector are given as

$$\binom{N-2b+s}{a}$$

for $0 \leq s \leq a$. We want to find for what value of s will this binomial coefficient be non-zero. Therefore, it follows that

$$\begin{aligned} N-2b+s &\geq a \\ a+b+1-2b+s &\geq a \\ s &\geq b-1. \end{aligned}$$

Thus, the value $s = b-1$ will imply that the binomial coefficient is positive. Hence,

$$\binom{N-2b+b-1}{a} = \binom{N-b-1}{a} = \binom{a}{a} = 1$$

is the first non-zero index of the ab-vector for the graph $D_N(a, N-b)$.

Now for the aw-vector. Recall that the indices for the ab-vector are

$$\binom{N-2a+s}{b},$$

for s between from 0 and b. We want to find for what values of s this binomial coefficient will be non-zero. Therefore,

$$\begin{aligned} N-2a+s &\geq b \\ a+b+1-2a+s &\geq b \\ s &\geq a-1 \end{aligned}$$

and $s = a-1$ will give the leading index. Thus,

$$\binom{N-2a+a-1}{b} = \binom{N-a-1}{b} = \binom{b}{b} = 1$$

is the leading index of the aw-vector for the graph $D_N(a, N-b)$.

We claim that the leading index of the ab-vector is larger than that of the aw-vector of the wedge product. Clearly the leading index for the ab-vector of the wedge product is given as

$$\binom{N-b-1}{a}\binom{M-2d}{c} \tag{20}$$

and the leading index for the aw-vector as

$$\binom{N-a-1}{b}\binom{M-2c}{d}. \tag{21}$$

Substituting in values for N, M, a, b, c, and d, expressions (20) and (21) become

$$\begin{aligned}\binom{N-b-1}{a}\binom{M-2d}{c} &= \binom{N-(N-(k+1))-1}{k}\binom{r+(r+1)l-2l}{l+1} \\ &= \binom{k}{k}\binom{r+rl-l}{l+1} \\ &= \binom{r+rl-l}{l+1}\end{aligned}$$

and

$$\begin{aligned}\binom{N-a-1}{b}\binom{M-2c}{d} &= \binom{r+(r+1)k-k-1}{r(k+1)-1}\binom{r+(r+1)l-2(l+1)}{l} \\ &= \binom{r+rk-1}{r+rk-1}\binom{r+rl-l-2}{l} \\ &= \binom{r+rl-l-2}{l}\end{aligned}$$

respectively. Let $A = r + rl - l$. First, notice that we can write

$$\binom{A}{l+1} = \binom{A-1}{l+1} + \binom{A-1}{l}$$

and we can also write

$$\binom{A-1}{l} = \binom{A-2}{l} + \binom{A-2}{l-1}.$$

Therefore,

$$\binom{A}{l+1} = \binom{A-1}{l+1} + \binom{A-2}{l-1} + \binom{A-2}{l}.$$

Now, because we have already shown that $\binom{A-2}{l} > 0$, it must be the case that $A - 2 \geq l$. But this implies that $A - 1 \geq l + 1$ and $A - 2 \geq l - 1$, which shows that each of these binomial coefficients are all positive (since $l \geq 1$). Therefore,

$$\begin{aligned}\binom{A}{l+1} &= \binom{A-1}{l+1} + \binom{A-2}{l-1} + \binom{A-2}{l} \\ &\geq 2 + \binom{A-2}{l} \\ &> \binom{A-2}{l}.\end{aligned}$$

It then follows that

$$\binom{r+rl-l}{l+1} \geq \binom{r+rl-l-2}{l},$$

as we wanted to show. Therefore, the first non-zero index of the *ab*-vector is larger than that of the *aw*-vector. □

One thing to notice here is that this proof does not rely on the fact that $l > k$. It still follows that the first non-zero index of the ab-vector is larger than that of the aw-vector even if $k > l$.

Corollary 1. *The graph $D(a, N-b, c, d; N, M)$ is semi-symmetric.*

Because the two Ivanov vectors are unequal, it follows that $D(a, N-b, c, d; N, M)$ is regular and strictly bi-transitive, hence showing semi-symmetry.

2.3 Third Construction: $D(a, b, M-c, d; N, M)$

If we construct this wedge product in the same way as $D(a, b, c, d; N, M)$, but this time, replace c with $M-c$, and we see that we satisfy the second construction for regularity given in Equation (7). It therefore follows that the graph $D(a, b, M-c, d; N, M)$ is regular and bi-transitive. Also, we see that the following relationships hold for $D(a, b, M-c, d; N, M)$:

$$a + rb = N \qquad \text{and} \qquad c + d + 1 = M.$$

We will now show that this construction yeilds a family of semi-transitive graphs.

Lemma 3. *The first non-zero index of the ab-vector of $D(a, b, M-c, d; N, M)$ occurs at*

$$\binom{N-2b}{a}. \tag{22}$$

Proof. The leading index of the ab-vector of $D_N(a, b)$ is non-zero (see Theorem 4) and is given by

$$\binom{N-2b}{a}.$$

Recall that the indices of the ab-vector of $D_M(M-c, d)$ are

$$\binom{M-2d+s}{c}$$

for s between 0 and c The leading index will the the smallest s such that

$$M - 2d + s \geq c.$$

Substituting in the relationships from above, we see that this occurs when $s \geq d-1$. Therefore, the smallest non-zero index in the ab-vector of $D_M(M-c, d)$ is

$$\binom{M-d-1}{c} = \binom{c}{c} = 1.$$

Therefore, by definition of the ab-vector of the graph $D(a, b, M-c, d; N, M)$ we see that the leading index is

$$\binom{N-2b}{a}.$$

□

Lemma 4. *The first non-zero index of the aw-vector of $D(a, b, M - c, d; N, M)$ occurs at*

$$\binom{N-2a}{b}. \tag{23}$$

Proof. The leading index of the aw-vector of $D_N(a, b)$ is non-zero (see Theorem 4) and is

$$\binom{N-2a}{b}.$$

Recall that the indices of the aw-vector of $D_M(M - c, d)$ are

$$\binom{M-2c+s}{d}$$

for s between 0 and a. The first (and smallest) non-zero index will the the smallest s that satisfies

$$M - 2c + s \geq d.$$

Substituting in the relationships from above, we see that this occurs when $s \geq c - 1$. Therefore, the leading index in the aw-vector of $D_M(M - c, d)$ is

$$\binom{M-c-1}{d} = \binom{d}{d} = 1.$$

Therefore, by definition of the aw-vector of the graph $D(a, b, M - c, d; N, M)$ we see that the leading index is

$$\binom{N-2a}{b}.$$

□

Theorem 7. *The leading index of the aw-vector is larger than in the ab-vector in the graph $D(a, b, M - c, d; N, M)$.*

Proof. Substituting in the values for N, M, a, b, c and b into (22) and (23), we see that the value of the leading index of the ab-vector becomes

$$\begin{aligned}\binom{N-2b}{a} &= \binom{r+(r+1)k-2(k+1)}{k}\\ &= \binom{r+rk-k-2}{k}\end{aligned}$$

and the value of the leading index of the aw-vector becomes

$$\begin{aligned}\binom{N-2a}{b} &= \binom{r+(r+1)k-2k}{k+1}\\ &= \binom{r+rk-k}{k+1}.\end{aligned}$$

Let $B = r + rk - k$. Notice that we can write

$$\binom{B}{k+1} = \binom{B-1}{k+1} + \binom{B-1}{k},$$

and we can also write

$$\binom{B-1}{k} = \binom{B-2}{k} + \binom{B-2}{k-1}.$$

Therefore, we have

$$\binom{B}{k+1} = \binom{B-1}{k+1} + \binom{B-2}{k} + \binom{B-2}{k-1}.$$

Because we have already shown that $\binom{B-2}{k} > 0$, it must be the case that $B-2 \geq k$. But this implies that $B - 1 \geq k + 1$ as well as $B - 2 \geq k - 1$, which shows that each of these binomial coefficients are positive, as $k \geq 1$. Therefore,

$$\begin{aligned}\binom{B}{k+1} &= \binom{B-1}{k+1} + \binom{B-2}{k} + \binom{B-2}{k-1} \\ &\geq 2 + \binom{B-2}{k} \\ &> \binom{B-2}{k}.\end{aligned}$$

It follows that

$$\binom{r+(r+1)k-2k}{k+1} > \binom{r+rk-k-2}{k}$$

as we wanted to show. Therefore, the first non-zero index of the aw-vector is larger than that of the ab-vector in the graph $D(a, b, M - c, d; N, M)$. □

Here, we sketch an alternative proof of the previous theorem.

Proof. Theorem 6 provides us with a nice tool. The proof of this theorem does not rely on the fact that $l > k$. What we will do is now assume the following: $r \geq 2, l \geq 1$, and $k > l$. Consider the graph $D(a, N - b, c, d; N, M)$. By making these assumptions, what we have essentially done, is changed the roles of l and k.

Switch l with k. That is, assume $k > l$. We might be able to think of this as swapping l with k. Again, the roles of l and k have been reversed. Now we flip the vertices around. The white vertices become black, while the black vertices become white. By flipping these vertices around, we have just changed the roles of the ab- and aw-vectors. What we have done, in essence, is just redefined $N = r + (r + 1)l$, $a = N - (l + 1)$, and $b = l$. We also have redefined $M = r + (r + 1)k$, $c = k$, and $d = k + 1$. Our graph now looks like $D_M(M - c, d) \wedge D_N(a, b) \cong D_N(a, b) \wedge D_M(M - c, d)$ but with the black and white vertices flip-flopped. So the ab- and aw-vectors switch as well, and we have proved the previously stated theorem. □

Corollary 2. *The graph $D(a, b, M - c, d; N, M)$ is semi-symmetric.*

2.4 Fourth Construction : $D(a, N-b, M-c, d; N, M)$

We construct our final wedge product in the same way as $D(a, b, c, d; N, M)$, but this time, replace b with $N-b$ and c with $M-c$, and we see that we satisfy the fourth construction for regularity given by Equation (8). It therefore follows that the graph $D(a, N-b, M-c, d; N, M)$ is regular and bi-transitive. Also, we see that the following relationships hold for $D(a, N-b, M-c, d; N, M)$:

$$a + b + 1 = N \qquad \text{and} \qquad c + d + 1 = M.$$

We will now show that this construction yeilds a family of semi-transitive graphs.

Lemma 5. *In the graph $D(a, N-b, M-c, d; N, M)$, the first non-zero index of both the ab- and aw-vectors is* 1.

Proof. Consider first the graph $D_N(a, N-b)$. The first non-zero index of the ab-vector is the smallest value s such that

$$\binom{N-2b+s}{a}$$

is positive. All we need then is

$$\begin{aligned} N - 2b + s &\geq a \\ a + b + 1 - 2b + s &\geq a \\ s &\geq b - 1. \end{aligned}$$

Therefore, the first non-zero index of the ab-vector occurs when $s = b - 1$, and so this index is

$$\binom{N-2b+b-1}{a} = \binom{a}{a} = 1.$$

The first non-zero index of the aw-vector is the smallest value s such that

$$\binom{N-2a+s}{b}$$

is positive. All we need then is

$$\begin{aligned} N - 2a + s &\geq b \\ a + b + 1 - 2a + s &\geq b \\ s &\geq a - 1. \end{aligned}$$

Therefore, the first non-zero index of the ab-vector occurs when $s = a - 1$, and so this index is

$$\binom{N-2a+a-1}{b} = \binom{b}{b} = 1.$$

Similar arguments will show that the first non-zero indices of both the ab- and aw-vectors of $D_M(M-c, d)$ are 1 as well.

Therefore, it clearly follows that the first non-zero indices of both the ab- and aw-vectors of the wedge product is also 1. □

Notice here that the first non-zero index of each vector is also the critical index of each vector.

Theorem 8. *In $D(a, N-b, M-c, d; N, M)$, the second non-zero index is larger in the aw-vector than it is in the ab-vector.*

Proof. Notice that the second non-zero index of the ab-vector (and aw-vector for that matter) for both $D_N(a, N-b)$ and $D_M(M-c, d)$ are the last index in each vector. Consider the last index of the ab-vector of $D_N(a, N-b)$. This index is

$$\binom{N-2b+b}{a} = \binom{a+1}{a} = a+1.$$

Also consider the last index of the aw-vector of $D_N(a, N-b)$. This index is

$$\binom{N-2a+a}{b} = \binom{b+1}{b} = b+1.$$

Notice that because $a+1 = k+1$ and $b+1 = r(k+1)$, it follows that the last index of the aw-vector of $D_N(a, N-b)$ is larger than the last index of the ab-vector.

Now, consider the last index of the ab-vector of $D_M(M-c, d)$. This index is

$$\binom{M-2d+d}{c} = \binom{c+1}{c} = c+1.$$

Also consider the last index of the aw-vector of $D_M(M-c, d)$. This index is

$$\binom{M-2c+c}{d} = \binom{d+1}{d} = d+1.$$

Notice that because $c+1 = r(l+1)$ and $d+1 = l+1$, it follows that the last index of the ab-vector of $D_M(M-c, d)$ is larger than the last index of the aw-vector.

Notice that the first non-zero index of the wedge product will be 1. This is because it is the product of the smallest two non-zero indices of each of the ab-vectors. The second non-zero index of the ab-vector of the wedge product will be the smallest of the two last indices of both ab-vectors. Since that index is the product of 1 and the last index, it follows that the second non-zero index of the ab-vector of the wedge product is just the smallest of the two. So all that we need to do is compare $a+1$ with $c+1$. Which one is smaller?

Recall that $l > k$. Then

$$\begin{aligned} c+1 &= r(l+1) \\ &> k+1 \\ &= a. \end{aligned}$$

Therefore, the second non-zero index of the ab-vector of the wedge product is $a+1$.

We do the same to determine the second non-zero index of the aw-vector of the wedge product. It will be the smallest of the last index of the two aw-vectors. All we do are compare $b+1$ with $d+1$. But here is where it doesn't matter which one is smaller. If $b+1$ is the smallest, notice that

$$b+1 = r(k+1) > k+1 = a+1.$$

If $d+1$ is the smallest, then

$$d+1 = l+1 > k+1 = a+1.$$

The thing to notice here is that in either case, the second non-zero index of the ab-vector of the wedge product is the smaller than the second non-zero index of the aw-vector of the wedge product. □

2.5 A New Construction of Normal Wedge Products

Looking at Pascal's Triangle some more, we see that Lemma 2 will generalize to the following, with the ratio between two consecutive entries being rational instead of just an integer.

Lemma 6. *For $p, q \in N$, $p, q \neq 0$, and $k \geq 0$, the following holds:*

$$\frac{\binom{(p+q-1)+(p+q)k}{q(k+1)}}{\binom{(p+q-1)+(p+q)k}{q(k+1)-1}} = \frac{p}{q}.$$

Proof. Using the factorial form of the binomial coefficient, we see that

$$\begin{aligned}\frac{\binom{(p+q-1)+(p+q)k}{q(k+1)}}{\binom{(p+q-1)+(p+q)k}{q(k+1)-1}} &= \frac{(p+q-1+(p+q)k)!}{(q(k+1))!(k(p+1)-1)!} \cdot \frac{(q(k+1)-1)!((p(k+1))!}{(p+q-1+(p+q)k)!} \\ &= \frac{(q(k+1)-1)!(p(k+1))!}{(q(k+1))!(p(k+1)-1)!} \\ &= \frac{p(k+1)}{q(k+1)} \\ &= \frac{p}{q}.\end{aligned}$$

□

Notice here that when $q=1$, we get the statement of Lemma 2. Notice that if $q < p$, then the entries we get are from the left half of Pascal's Triangle. If instead $p < q$, then we are just looking at the other side of Pascal's Triangle, and get ratios less than one. I don't think that we have to restrict $q < p$.

Notice that if $q < p$, then $q(k+1) - 1 < \frac{p+q-1+(p+q)k}{2}$. This fact re-iterates the comment that if $q < p$, then the entries are from the left half of Pascal's Triangle.

Using this lemma to construct regular wedge products works the same as before, except that the proofs of semi-symmetry of such graphs will fall into about four cases for each construction. This gets a bit tedious, so instead, we go back to the begining, and consider under what circumstances the condition of regularity will imply semi-symmetry.

For the following constructions, let $N = (p+q-1)+(p+q)k$, $a = q(k+1)-1$, and $b = q(k+1)$. For $l > k$, let $M = (p+q-1)+(p+q)l$, $c = q(l+1)$, and $d = q(l+1)-1$.

3 Back to the Begining

Consider the graph $D(a, b, c, d; N, M)$ and assume that the condition of regularity holds. That is, assume that $a < b < N$, $d < c < M$, and following equation is satisfied:

$$\binom{N}{b}\binom{M}{d} = \binom{N}{a}\binom{M}{c}.$$

If we expand this equation, we get

$$\frac{a!(N-a)!}{b!(N-b)!} = \frac{d!(M-d)!}{c!(M-c)!}. \tag{24}$$

Notice here that we no longer require anything to be less than $N/2$ or $M/2$.

Consider the critical index of the ab-vector. By Theorem 2, expanding the ab-vectors ($a1b$ and $a2b$) of both $D_N(a, b)$ and $D_M(c, d)$, we get

$$a1b = \ldots + \left\{\binom{N-b-1}{a}, b(N-b)\right\} + \left\{\binom{N-b}{a}, 1\right\} \text{ and}$$
$$a2b = \ldots + \left\{\binom{M-d-1}{c}, d(N-d)\right\} + \left\{\binom{M-d}{c}, 1\right\}$$

Then by Theorem 3, this index is

$$b_1 = \binom{N-b}{a}\binom{M-d-1}{c}$$

or

$$b_2 = \binom{N-b-1}{a}\binom{M-d}{c},$$

whichever is larger. Likewise, we see that the critical index of the aw-vector is

$$w_1 = \binom{N-a}{b}\binom{M-c-1}{d}$$

or

$$w_2 = \binom{N-a-1}{b}\binom{M-c}{d},$$

whichever is larger.

3.1 Ratios of Possible Critical Indices

We will now study the ratios of how b_1, b_2, w_1, and w_2 relate to eachother.

Comparing b_1 with w_1, we get

$$\begin{aligned}\frac{b_1}{w_1} &= \frac{\binom{N-b}{a}\binom{M-d-1}{c}}{\binom{N-a}{b}\binom{M-c-1}{d}}\\ &= \frac{b!(N-b)!}{a!(N-a)!}\cdot\frac{d!(M-d-1)!}{c!(M-c-1)!}\\ &= \frac{b!(N-b)!}{a!(N-a)!}\cdot\frac{d!(M-d)!}{c!(M-c)!}\cdot\frac{M-c}{M-d}.\end{aligned}$$

Recall that the assumed condition of regularity implies Equation 24. Solving for 1 and substituting into the expression above, we see that

$$\frac{b_1}{w_1} = \frac{M-c}{M-d}.$$

Since $c > d$, it follows that $M - c < M - d$, and hence it must be the case that $w_1 > b_1$.

Compare b_2 with w_2, and using the regularity assumption, we get

$$\begin{aligned}\frac{b_2}{w_2} &= \frac{\binom{N-b-1}{a}\binom{M-d}{c}}{\binom{N-a-1}{b}\binom{M-c}{d}}\\ &= \frac{b!(N-b-1)!}{a!(N-a-1)!}\cdot\frac{d!(M-d)!}{c!(M-c)!}\\ &= \frac{N-a}{N-b}.\end{aligned}$$

Since $b > a$, it follows that $N - a > N - b$, and hence it must be the case that $b_2 > w_2$.

Compare b_1 with b_2:

$$\begin{aligned}\frac{b_1}{b_2} &= \frac{\binom{N-b}{a}\binom{M-d-1}{c}}{\binom{N-b-1}{a}\binom{M-d}{c}}\\ &= \frac{(N-b)!}{a!(N-b-a)!}\cdot\frac{(M-d-1)!}{c!(M-c-d-1)!}\cdot\frac{a!(N-b-a-1)!}{(N-b-1)!}\cdot\frac{c(M-c-d)!}{(M-d)!}\\ &= \frac{(N-b)!}{(N-b-1)!}\cdot\frac{(N-b-a-1)!}{(N-b-a)!}\cdot\frac{(M-d-1)!}{(M-d)!}\cdot\frac{(M-c-d)!}{(M-c-d-1)!}\\ &= \frac{(N-b)(M-c-d)}{(N-a-b)(M-d)}.\end{aligned}$$

After some algebra, we find that $\frac{N-b}{a} < \frac{M-d}{c}$ implies $b_1 > b_2$ (and visa-versa).

Comparing w_1 with w_2, we see that

$$\frac{w_1}{w_2} = \frac{(N-a)(M-c-d)}{(M-c)(N-a-b)}.$$

After some algebra, we find that $\frac{N-a}{b} < \frac{M-c}{d}$ implies $w_1 > w_2$ (and visa-versa).

Compare b_1 with w_2, and using expression 24,

$$\begin{aligned}
\frac{b_1}{w_2} &= \frac{\binom{N-b}{a}\binom{M-d-1}{c}}{\binom{N-a-1}{b}\binom{M-c}{d}} \\
&= \frac{b!(N-b)!}{a!(N-a-1)!} \cdot \frac{d!(M-d-1)!}{c!(M-c)!} \cdot \frac{(N-a-b-1)!(M-c-d)!}{(N-a-b)!(M-c-d-1)!} \\
&= \frac{N-a}{M-d} \cdot \frac{M-c-d}{N-a-b}.
\end{aligned}$$

It then follows (through some algebra) that $\frac{N-a}{b} > \frac{M-d}{c}$ implies that $b_1 > w_2$ (and visa-versa).

Compare b_2 with w_1:

$$\begin{aligned}
\frac{b_2}{w_1} &= \frac{\binom{N-b-1}{a}\binom{M-d}{c}}{\binom{N-a}{b}\binom{M-c-1}{d}} \\
&= \frac{b!(N-b-1)!}{a!(N-a)!} \cdot \frac{d!(M-d)!}{c!(M-c-1)!} \cdot \frac{(N-a-b)!(M-c-d-1)!}{(N-a-b-1)!(M-c-d)!} \\
&= \frac{M-c}{N-b} \cdot \frac{N-a-b}{M-c-d}.
\end{aligned}$$

It then follows that $\frac{N-b}{a} > \frac{M-c}{d}$ implies that $b_2 > w_1$ (and visa-versa).

So, to wrap things up, under the assumption that $\binom{N}{b}\binom{M}{d} = \binom{N}{a}\binom{M}{c}$, we have the following:

$$\begin{aligned}
& & w_1 &> b_1, \\
& & b_2 &> w_2, \\
\frac{N-b}{a} > \frac{M-d}{c} &\iff & b_2 &> b_1, \\
\frac{N-a}{b} > \frac{M-c}{d} &\iff & w_2 &> w_1, \\
\frac{N-a}{b} > \frac{M-d}{c} &\iff & w_2 &> b_1, \text{ and} \\
\frac{N-b}{a} > \frac{M-c}{d} &\iff & b_2 &> w_1.
\end{aligned}$$

Notice here that the sixth condition, along with the given conditions 1 and 2, is necessary to place an ordering on the possible critical indices, and hence determine *the* critical index. If we have inequality in the fractions $\frac{N-b}{a}$ and $\frac{M-c}{d}$, we have guaranteed that the critical indices are different. What was just developed in this section is enough to prove that all four constructions before to be semi-symmetric. However, to prove this statement, we will instead prove that the more general construction baseds on Lemma 6 (of which Lemma 2 is a special case) are semi-symmetric, using either condition 6, or one of either conditions 3 and 4.

4 Results on Semi-Symmetry of Previous Graphs

4.1 Construction 1: $D(a, b, c, d; N, M)$

We will show $\frac{N-b}{a} > \frac{M-c}{d}$, showing that b_2 is the critical index. Assume to the contrary that $\frac{N-b}{a} \le \frac{M-c}{d}$. Then,

$$\begin{aligned}
\frac{N-b}{a} &\le \frac{M-c}{d} \\
\frac{(p+q-1+pk+qk)-(qk+k)}{qk+q-1} &\le \frac{(p+q-1+pl+ql)-(ql+q)}{ql+q-1} \\
\frac{p+pk-1}{qk+q-1} &\le \frac{p+pl-1}{ql+q-1} \\
(p+pk-1)(ql+q-1) &\le (p+pl-1)(qk+q-1) \\
-pk-ql &\le -pl-qk \\
qk-ql &\le pk-pl \\
q(k-l) &\le p(k-l) \\
q(l-k) &\ge p(l-k)
\end{aligned}$$

which is a contradiction, since $p > q$. Therefore it follows that $\frac{N-b}{a} > \frac{M-c}{d}$, so condition 6 holds and we see that the critical indices are different, thus showing semi-symmetry in the wedge product.

4.2 Construction 2: $D(a, N-b, c, d; N, M)$

We will show that condition 6 holds, by assuming to the contrary that $\frac{N-b}{a} = \frac{M-c}{d}$ and then proving that this cannot be the case.

$$
\begin{aligned}
\frac{N-b}{a} &= \frac{M-c}{d} \\
\frac{qk+q}{qk+q-1} &= \frac{p+pl-1}{ql+q-1} \\
\frac{qk+q}{qk+q-1} &= \frac{p(l+1)-1}{q(l+1)-1} \\
(qk+q)(q(l+1)-1) &= (qk+q-1)(p(l+1)-1) \\
q(qk+q)(l+1) - p(qk+q)(l+1) + p(l+1) &= 1 \\
(l+1)\big(q(qk+q) - p(qk+q) + p\big) &= 1.
\end{aligned}
$$

Notice that since$(l+1)\big(q(qk+q) - p(qk+q) + p\big)$ is divisible by $l+1$, it also follows that 1 must also be divisible by $l+1$. However, this only happens when $l = 1$. But since we assumed that $l > k > 0$, it follows that $l > 1$, and we have a contradiction. Therefore, the wedge vectors are different, showing semi-symmetry.

4.3 Construction 3: $D(a, b, M-c, d; N, M)$

We will show that condition 6 holds, by assuming to the contrary that $\frac{N-b}{a} = \frac{M-c}{d}$ and then proving that this cannot be the case.

$$
\begin{aligned}
\frac{N-b}{a} &= \frac{M-c}{d} \\
\frac{p+pk-1}{q+qk-1} &= \frac{ql+q}{ql+q-1} \\
\frac{p(k+1)-1}{q(k+1)-1} &= \frac{ql+q}{ql+q-1} \\
(ql+q-1)(p(k+1)-1) &= (ql+q)(q(k+1)-1) \\
p(ql+q)(k+1) - (ql+q) - p(k+1) + 1 &= q(ql+q)(k+1) - (ql+q) \\
q(ql+q)(k+1) + p(k+1) - p(ql+q)(k+1) &= 1 \\
(k+1)\big(q(ql+q) - p(ql+q) + p\big) &= 1.
\end{aligned}
$$

Notice that since $(k+1)\big(q(ql+q) - p(ql+q) + p\big)$ is divisible by $k+1$, it follows that 1 must also be divisible by $k+1$. This only happens when $k = 0$. So assume that $k = 0$. Then we get

$$\begin{aligned}
\frac{N-b}{a} &= \frac{M-c}{d} \\
\frac{p-1}{q-1} &= \frac{ql+q}{ql+q-1} \\
(p-1)(ql+q-1) &= (q-1)(ql+q) \\
pql+pq-p-ql-q+1 &= q^2l+q^2-ql-q \\
p(ql+q-1)+1 &= q(ql+q-1) \\
(p-q)(ql+q-1) &= -1.
\end{aligned}$$

Since $p-q > 0$ and $ql+q-1 > l > 1$, it follows that $(p-q)(ql+q-1) > -1$, hence it cannot be the case that $\frac{N-b}{a} = \frac{M-c}{d}$. Therefore, the black and white vectors of the wedge product are different, showing semi-symmetry.

4.4 Construction 4: $D(a, N-b, M-c, d; N, M)$

We will show that $\frac{N-b}{a} > \frac{M-c}{d}$. Assume to the contrary that $\frac{N-b}{a} \leq \frac{M-c}{d}$. Then,

$$\begin{aligned}
\frac{N-b}{a} &\leq \frac{M-c}{d} \\
\frac{qk+q}{q+qk-1} &\leq \frac{ql+q}{q+ql-1} \\
(qk+q)(q+ql-1) &\leq (ql+q)(q+qk-1) \\
q^2kl+q^2k-qk+q^2l+q^2-q &\leq q^2kl+q^2l-ql+q^2k+q^2-q \\
-qk &\leq -ql \\
ql &\leq qk
\end{aligned}$$

which is not the case, since $l > k$. Therefore, it follows that $\frac{N-b}{a} > \frac{M-c}{d}$, showing that the critical indices are different, and hence the graph is semi-symmetric.

5 Conclusions

A number of open questions still remain. Notice that the condition $\frac{N-b}{a} = \frac{M-c}{d}$ implies that the critical indices of both the ab- and aw-vectors are the same. However, this does not necessarily imply that two vectors are the same (we have yet to find evidence of this, though). Does equality of the critical indices imply that the graph is not semi-symmetric? But what are the conditions to show that the two vectors are identical? One could decide to look at the length of both vectors to gain additional information. Also, what about the entry at the location of the critical index? Are they the same for both vectors? What additional information will decide this?

Another question to consider is that concerning the wedge product of more than two $D_N(a,b)$'s. Under what conditions will the product of three such graph be semi-symmetric? What about the product of four graphs? n graphs?

Finally, the lemmas that we used to construct regular products of the D_N's relied on the fact that the two entries in Pascal's Triangle were consecutive. Can anything be predicted about the ratios of non-consecutive entries in the triangle? This is the topic of my future research.

References

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