Blowing Up the Braid Arrangement

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A central hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ in $\mathbb{C}^l$ is a collection of linear subspaces H_i of codimension 1. If the hyperplanes of $\mathcal{A}$ share a point in common then we call $\mathcal{A}$ a central arrangement. We aim to study the topological properties of the complement $\mathbf{M} = \mathbb{C}^l - \mathcal{A}$. In particular we wish to calculate the local system cohomology $H^*(M, \mathcal{L}_\lambda)$ determined by weight vector $\lambda \in \mathbb{C}^n$. The i^{th} component λ_i of λ is assigned to the hyperplane H_i for each $i = 1, \dots, n$.

Let $\{\varphi_1, \dots, \varphi_n\}$ be the set of linear forms associated with the hyperplanes in $\mathcal{A}$. Then for each $i = 1, \dots, n$ we have $\ker \varphi_i = H_i$. Define $\omega_i = d\varphi_i/\varphi_i$ for $i = 1, \dots, n$. Let $C^*(M)$ be the ring of complex differential forms on M. Set $\omega_\lambda = \sum_{i=1}^n \lambda_i \omega_i$. Define $\nabla_\lambda : C^k(M) \to C^{k+1}(M)$ by $\nabla_\lambda(\tau) = d\tau + \omega_\lambda \wedge \tau$. A quick calculation shows that $\nabla_\lambda^2 = 0$ and we therefore obtain the chain complex

$$0 \to C^0(M) \to C^1(M) \to C^2(M) \to \cdots \tag{1}$$

Let $B^*(M) = \langle \omega_1, \dots, \omega_n \rangle$ be a subring of $C^*(M)$. Then there is an isomorphism $\eta : B^*(M) \to H^*(M)$ defined by $\eta(\omega_i) = [\omega_i]$. The function η takes the logarithmic differential form associated with each hyperplane to its corresponding cohomology class. Thus $B^*(M) \cong H^*(M)$.

To each hyperplane H_i in $\mathcal{A}$ assign the symbol e_i. Form the exterior algebra $E = \Lambda(e_1, \dots, e_n)$. For $S = \{i_1, \dots, i_p\} \subseteq [n] = \{1, \dots, n\}$, let e_S denote the product $e_{i_1} \wedge \cdots \wedge e_{i_p}$, where $i_1 < \cdots < i_p$. We can define a rank function $r : [n] \to \mathbb{Z}^+$ by $r(S) = \operatorname{codim}(\bigcap_{i \in S} H_i)$. We call S dependent if $r(S) < |S|$. We write $\bigcap S = X$ if $\bigcap_{i \in S} H_i = \bigcap_{i \in X} H_i$. A flat $X \subseteq [n]$ is a maximal dependent set. We can form the lattice of flats L for the arrangement $\mathcal{A}$, ordered by reverse inclusion. Thus if $X, Y \in L$ then $X \leq Y$ if and only if $Y \subseteq X$. If the arrangement $\mathcal{A}$ is central, then L has a unique maximal element $1 = \bigcap_{i=1}^n H_i$. The minimal element 0 of L is the the entire space $\mathbb{C}^l$. Ordered in this fashion L satisfies the requirements for being a geometric lattice. Now let I be an ideal of E generated by the set $\{\partial e_S \,|\, r(S) < |S|\}$, where ∂ is the usual boundary operator given by $\partial e_S = \sum_{k=1}^p (-1)^{p-1}(e_{i_1} \wedge \cdots e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots e_{i_p})$. For convenience denote $e_{i_1} \wedge \cdots e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots e_{i_p}$ by e_{S_k}. The quotient $A = E/I$ is called the Orlik-Solomon algebra (OS-algebra) for the arrangement $\mathcal{A}$.

Immediately we see that the OS-Algebra A can be coarsely graded by $[n]$: $A = \bigoplus_{p=0}^n A^p$. It is a well known result that the OS-Algebra A can be given a finer grading by the lattice of flats L of $\mathcal{A}$. For each $X \in L$, let $E_X = \text{span}\{e_S \mid \bigcap S = X\}$. Let $\pi : E \to A$ be the natural map and set $A_X = \pi(E_X)$. Then we have $A = \bigoplus_{X \in L} A_X$. Furthermore, it is also true that $A^p = \bigoplus_{x \in L, r(X)=p} A_X$. As a result of this fact and a lemma due to Brieskorn, we have an isomorphism of graded algebras $H^*(M) \cong A$. Define $a_\lambda : E^k \to E^{k+1}$ by left multiplication by $a_\lambda = \sum_{i=1}^n \lambda_i e_i$. Restricting a_λ to A we obtain the chain complex

$$0 \to A^1 \to A^2 \to A^3 \to \cdots \tag{2}$$

When an arrangement $\mathcal{A}$ has connected flats X in the associated lattice of flats $L = L(\mathcal{A})$ that have the property $\lambda_X = \sum_{i \in X} \lambda_i = 0$ we can no longer apply some big fancy theorems to tell us what the dimensions of the cohomology groups for M are. This is the motivation for the technique of blowing up the arrangement at bad connected flats. Once blown up, the flat is no longer a flat in the sense that the hyperplanes of the flat no longer intersect. In blowing up we change the topology of the complement only locally around the blow-up point and everything else remains the same. This does have the effect of changing the presentation for the cohomology groups of the complement, but we should, hopefully, be able to easily calculate the dimensions of the original cohomology from the blown-up version, which is easier to handle.

Blowing Up in the Projective Plane:

Each point p in the projective plane can be written as the ordered pair $\{(x, y), [S : T]\}$ where (x, y) denotes the x and y coordinates of p on the projective plane and $[S : T]$ denotes the line l which passes through the origen and contains the point p. For $(x, y) \neq (0, 0)$ the relation $xT = Sy$ holds. Consider the set U consisting of the points on the projective plane not equal to $(0, 0)$. There exist subsets $\mathcal{U}_0$ and $\mathcal{U}_1$ of U where $\mathcal{U}_0 = \{(x, y), [S : T] \mid S \neq 0\}$ and $\mathcal{U}_1 = \{(x, y), [S : T] \mid T \neq 0\}$. It follows that $\mathcal{U}_0 \cup \mathcal{U}_1 = U$. Since homogeneous coordinates $[S : T]$ are equal to $[cS : cT]$ for any constant c, then $\mathcal{U}_0$ and $\mathcal{U}_1$ can be re-written as $\mathcal{U}_0 = \{(x, y), [1 : t] \mid t = \frac{T}{S},\ S \neq 0,\ xt = y\}$ and $\mathcal{U}_1 = \{(x, y), [s : 1] \mid s = \frac{S}{T},\ T \neq 0,\ x = sy\}$. For an 3-arrangement $\mathcal{A}$, consider the defining polynomial $Q(\mathcal{A}) = Q(x, y, z)$. In the projective plane, the arrangement we look at is $Q(x, y, 1)$. Define the **pullback function** π to be $\pi(\mathcal{A}) = Q(x, xt, 1)$. Substituting $x = ys$ we see that $\pi(\mathcal{A}) = Q(ys, y, 1)$ also, therefore $\pi(\mathcal{A})$ is defined everywhere on U. To **blow up** an arrangement $\mathcal{A}$ locally, we use the pullback function π. For example, $Q(\mathcal{A}) = xyz(x - y)$ blown up at the intersection point of the hyperplanes gives us $\pi(\mathcal{A}) = x^3t(1 - t)$ and $\pi(\mathcal{A}) = y^3s(s - 1)$. Since we're blowing up at $(x, y) = (0, 0)$, the hyperplanes $x = 0$ and $y = 0$ correspond to the **exceptional divisor** in the blow up. The hyperplanes $t = 0$ and $t = 1$ in the blown up space correspond respectively to the hyperplanes $y = 0$ and $x = y$ in projective space. Similarly, the hyperplanes $s = 0$ and $s = 1$ in the blown up space correspond respectively to the hyperplanes $x = 0$ and $x = y$ in projective space.

Since the cohomology of the complement is isomorphic to the OS-algebra of the arrangement, we can formulate a new presentation for the cohomology of the complement in a purely algebraic way as a presentation for a new, blown-up, OS-algebra which we denote by $\tilde{\mathrm{A}}$.

Definition: Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$ be the associated lattice of flats for the arrangement $\mathcal{A}$. Define a subset $\mathcal{B}$ of L by $\mathcal{B} = \{Z \in L \mid \lambda_Z = 0\}$.

The set $\mathcal{B}$ is the set of points in the lattice of flats $L(\mathcal{A})$ that are blown up. For each point in $\mathcal{B}$ we obtain a new set of relations in the presentation for the OS-algebra. We derived these new relations by considering the sequence of the pair in cohomology for the topological space obtained locally around a blow up point.

Definition: Let $\mathcal{A}$ be an arrangement and let A be the OS-algebra for $\mathcal{A}$. For each $Z \in \mathcal{B}$, let $u_Z = \sum_{i \in Z} e_i$. Define an ideal P in E by $P = \langle u_Z \mid Z \in \mathcal{B} \rangle$. Let $\tilde{\mathrm{A}}= A/(I+P)$, where I is the ideal of E generated by ∂e_S, S dependent. We call $\tilde{\mathrm{A}}$ the blown up Orlik-Solomon algebra.

We would like to be able to extend the theorems that apply to the generic OS-algebra to the blown up OS-algebra with the goal of showing that we have some sort of isomorphism of graded algebras, $H^*(\tilde{\mathrm{M}}) \cong \tilde{\mathrm{A}}$, where $H^*(\tilde{\mathrm{M}})$ is the cohomology for the complement of the blown up arrangement. In order to do this we have to find an appropriate grading of $\tilde{\mathrm{A}}$ by the intersection lattice. We looked at two approaches to this problem, both modelled after the theory developed for regular OS-algebras. The first approach is to think of the blown up OS-algebra as a quotient of the old OS-algebra over the ideal P, that is, $\tilde{\mathrm{A}}= A/P$. We know that $A = \bigoplus_{X \in L} A_X$ so if we can prove that $P = \bigoplus_{X \in L} P_X$, where $P_X = E_X \cap P$, then we would get what is needed.

Definition: Let $\pi_X : E \to E_X$ we the projection of E onto E_X defined by $\pi_X(e_S) = e_S$ if $\cap S = X$ and $\pi_X(e_S) = 0$ otherwise.

To show that $P = \bigoplus_{X \in L} P_X$, it suffices to show that $\pi_X(P) \subseteq P$ for every $X \in L$. But we quickly see that this cannot happen since if $Z \in \mathcal{B}$ and $i \in Z$ then $\pi_i(u_Z) = e_i$, which is not a member of the ideal P. Thus our niave approach to grading P is incorrect.

A second method for obtaining a grading for $\tilde{\mathrm{A}}$ is to use Gröbner basis theory. Here we give some background on the theory, which is usually formulated in terms of polynomial algebras, and explain how it can be extended to exterior algebras.

Let K be a field. Then $K[x_1, \ldots, x_n]$ is the polynomial algebra in n indeterminants with coefficients in K. A major theorem that allows the theory of Gröbner bases to work is the following:

Hilbert Basis Theorem. Every ideal I contained in $K[x_1, \ldots, x_n]$ is finitely generated.

This means that given any ideal $I \subseteq K[x_1, \ldots, x_n]$, we can find polynomials $g_1, \ldots, g_s$ such that $I = \langle g_1, \ldots, g_s \rangle$. It is possible to extend this result to exterior algebras. Thus any ideal I of E is finitely generated. In order to talk about G-bases we need to organize our polynomials. We do this be introducing an order on the terms of polynomials in $K[x_1, \ldots, x_n]$. For a vector $\alpha \in \mathbb{R}^n$, by x^α we mean $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The corresponding notation in exterior algebras has already been introduced; recall that for $S \subseteq [n]$ the element $e_S \in E$ represents the wedge product $\Pi_{i \in S} e_i$.

Definition: Let $c_\alpha x^\alpha$ and $c_\beta x^\beta$ be monomials in $K[x_1, \ldots, x_n]$. We say $c_\alpha x^\alpha < c_\beta x^\beta$ if $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, or $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and there is an index m, $1 \leq m \leq n$, such that $\alpha_i = \beta_i$ for all $i < m$ and $\alpha_m < \beta_m$.

For example, in variables x, y, z, we have $xy^2z < x^2yz$. This way of ordering monomials is usually called graded lexicographic order, or graded lex for short. There is an analogous definition for a graded lex ordering on monomials in the exterior algebra E.

Definition: Let e_T and e_S be monomials in E, where $S = \{i_1, \ldots, i_p\}$ and $T = \{j_1, \ldots, j_q\}$. We say $e_S < e_T$ if $p < q$ or if $p = q$ and there exists an index $m, 1 \leq m \leq p$, such that $i_k < j_k$ for all $k < m$ and $i_m < j_m$.

Definition: Let $f = \sum e_S \in E$. Then
(a) $\text{In}(f)$ is the largest term (initial term) in f with respect to the graded lex ordering.
(b) $\text{Im}(f)$ is the largest term of f without the coefficient (initial monomial).

Definition: For an ideal I of E, let $\text{In}(I)$ be the ideal generated by the set of initial terms of elements of I.

Since I is finitely generated, we have $\text{In}(I) = \langle\ \text{In}(g_1), \ldots, \text{In}(g_s)\ \rangle$ for some $g_1, \ldots, g_s \in E$. Now we can say what a G-basis for an exterior algebra ideal is.

Definition: A G-basis for an ideal I in E is a set of generating polynomials $G = \{g_1, \ldots, g_s\}$ for I such that $\text{In}(G) = \text{In}(I)$.

There is a nice algorithm called Buchberger's Algorithm that gives a way to test a given basis to see whether or not it is a G-basis. The algorithm relies on the division algorithm in $K[x_1, \ldots, x_n]$ and something called Syzygy polynomials. Since we are interested only in the application of this algorithm to exterior algebras, we will skip the usual presentation of Buchberger's algorithm in polynomial algebras and describe its implementation in exterior algebras. We will accept on faith that there is a division algorithm for polynomials in E. For two polynomials f and g in E, where $Im(f) = e_S$ and $Im(g) = e_T$ we can define their least common multiple (lcm) to be $e_{S \cup T}$

Definition: Let $f, g \in E$. Define the syzygy polynomial (S-polynomial) of f and g to be

$$s(f,g) = \frac{lcm(f,g)}{Im(f)} \cdot f + \frac{lcm(f,g)}{Im(g)} \cdot g. \tag{3}$$

For any polynomial f in E we define the jth T-polynomial of f to be $t_j(f) = e_j \cdot f$, where e_j is a factor of $Im(f)$.

In order to test a basis to see if it is a G-basis we need only check that each S and T polynomial for the elements in the basis yields a remainder of zero when divided by the elements of the basis. For the regular OS-Algebra, it is known that the set $\{\partial e_S \,|\, \text{S is dependent}\}$ is a G-Basis for the ideal I. A result from G-basis theory states that if the set G is a G-basis for the ideal I in E, then the linear complement of $\text{In}(G)$, call it C, is a basis for the quotient E/I. This gives an easy criterion for form a basis for the quotient, namely that no element of the basisi should be divisible by a memeber of In(G). This provides the motivation for finding a G-basis for $I + P$ as it allows the calculation of a basis for $E/(I + P)$ and its dimension. Thus we wanted to know whether or not the presentation we formulated for $I + P$ gave a G-basis for $I + P$. To check we calculated the S and T polynomials for pairs of elements in the presentation. But we found quite quickly that some S-polynomials did not have a remainder of zero upon division by elements of our basis.

To extend the theorems for the regular OS-Algebra to the blown up OS-Algebra we may have to recalculate our presentation for the blow up. If our current hypothesis is inaccurate, then it is wasteful to continue to try to find a grading for the blown up OS-Algebra using an incorrect presentation. These ideas remain an interesting problem in the connection between the algebra and topology of hyperplane arrangements.

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