FINDING SEMI-TRANSITIVE ORIENTATIONS OF GRAPHS

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Abstract. Given any graph, how do we determine whether it is semi-transitive or not. In this paper, I give a method to determine the semi-transitive orientations of a graph given that the graph is known to be semi-transitive. Further work will be needed to say that this definitely works in all cases, but I provide multiple examples that seem to show that this method does indeed produce all orientations of a semi-transitive graph. The reader is left with open questions and conjectures.

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1. Introduction

In this paper, all graphs and directed graphs are connected and simple.

Definition 1.1. *Let G be a group and X be a set. An* action *of G on X is a mapping of XxG to X such that*

(1) *for all $g, h \in X$, $x(gh) = (xg)h$.*
(2) *if I is the identity element of G, then $xI = x$ for all x in X*

In this paper, I will be mostly be concerned with the action of the Automorphism Group of a Graph on the vertices, edges, and darts of the graph.

Definition 1.2. *A* symmetry *(or* automorphism*) of a graph is a permutation of the graph's vertices that preserves edges.*

The symmetries of a graph Γ form a group $Aut(\Gamma)$, the Automorphism group of the graph, under composition.

Definition 1.3. *Let x be an element of a set X. The* orbit *of x is the set, $xG = \{xg|g \in G\}$.*

The orbit is the set of all things that x can be sent to by the elements of G. In this paper, we will mostly be concerned with the orbit of vertices, edges, and darts under the Automorphism Group of the graph and subsets of the Automorphism Group.

It is useful to note that the orbits partition the set X by the following lemma:

Lemma 1.4. *For any $x, y \in X$, y is in the orbit of $x \Leftrightarrow x$ is in the orbit of y.*

Definition 1.5. *The action of a group G on a set X is* transitive *provided that $\forall x, y \in X \exists g \in G$ such that $y = xg$.*

A group G is transitive on X provided that the action of G on X has only one orbit, X. So every element in X can be sent to any other element in X by some action of G.

For this paper I will be mostly be concerned with whether the Automorphism Group or a subset of the Automorphism Group is transitive on vertices, edges, or darts. Remember that a dart is a directed edge. In this paper I will denote an edge from the vertex u to the vertex v as $\{u, v\}$, and a directed edge from u to v as (u, v).

Definition 1.6. *The* stabilizer *of $x \in X$ in G is the set $G_x = \{g \in G | xg = x\}$.*

So the stabilizer of x is the set of elements from g that send x to x. If g is in the stabilizer of x, we say g fixes x.

Definition 1.7. *A Graph Γ is* semi-transitive *if there exists $G \subseteq Aut(\Gamma)$ such that G acts transitively on vertices and on edges, but not on darts.*

Definition 1.8. *Let d be one dart of a semi-transitive graph Γ. An* orientation *of Γ is the orbit dG. The orientation is a directed graph having one dart on each edge of Γ.*

So, semi-transitive graphs are vertex-transitive and edge-transitive on some subset of the Automorphism group of the Graph, but not dart-transitive.

Notice that if one vertex has out-degree d in the orientation, than every vertex has out-degree d and in-degree d in the orientation. Thus, the underlying graph Γ of the orientation is regular of degree $2d$. So a semi-transitive graph Γ is regular of even degree. This also results from the following theorem.

Theorem 1.9 (Tutte)**.** *If a graph Γ is regular of odd-degree, vertex-transitive and edge-transitive, then Γ is dart-transitive.*

An interesting theorem follows that gives a condition on when a graph is dart-transitive. Since I am looking for graphs that fail to be transitive on darts, it is important to know what to stay away from.

Theorem 1.10. *Suppose Γ is a graph, v a vertex, $G = Aut(\Gamma)$, and G_v is transitive on the set of neighbors of v. Then, G is dart-transitive.*

But neighbors are important when determining whether a digraph is a semi-transitive orientation of its underlying graph.

Theorem 1.11. *Let $\triangle$ be a digraph, $G = Aut(\triangle)$, and $v \in V(\triangle)$. Let*

(1) *$\triangle$ be vertex-transitive.*
(2) *G_v be transitive on the out-neighbors of v.*

Then, $\triangle$ is a semi-transitive orientation of the underlying graph $UG(\triangle)$.

Recall to get the underlying graph of a digraph you simply replace all directed edges with undirected edges. The resulting graph is the underlying graph of the digraph.

2. Semi-Transitive Orientations of Circulant Graphs

In this paper I am concerned with finding Semi-Transitive Orientations of graphs. Finding orientations of Circulant Graphs has already been exhaustively studied by Saner, Yuhasz, and Wilson. I studied circulant graphs and their methods in the hopes of generalizing the investigation of semi-transitive orientations to all graphs. I present some of the most interesting and relevant information about finding semi-transitive orientations of circulant graphs. I add one interesting result that I discovered in my study of circulant graphs.

Definition 2.1. *A* circulant graph *is a graph $C_N(S)$ with vertex set $\mathbb{Z}_N$ where S is a non-empty subset of the non-zero numbers mod N such that for each vertex i there is the edge $\{i, i+s\}$ where $s \in S$.*

The set S is often called the *jump set* of the graph since each vertex jumps that much to create an edge. Notice that if for each vertex i and every $s \in S$ there is the edge $\{i, i+s\}$, there there is also the edge $\{i, i-s\}$ in the graph.

We denote the *directed circulant graph* $C_N[S]$, where the edge $(i, i+s)$ goes from i to $i+s$ for every vertex i and for all $s \in S$. Clearly, $C_N(S)$ is the underlying graph of $C_N[S]$.

Example 2.2. *Below is the circulant graph $C_{12}(1,5)$ and the directed circulant graph $C_{12}[1,5]$.*

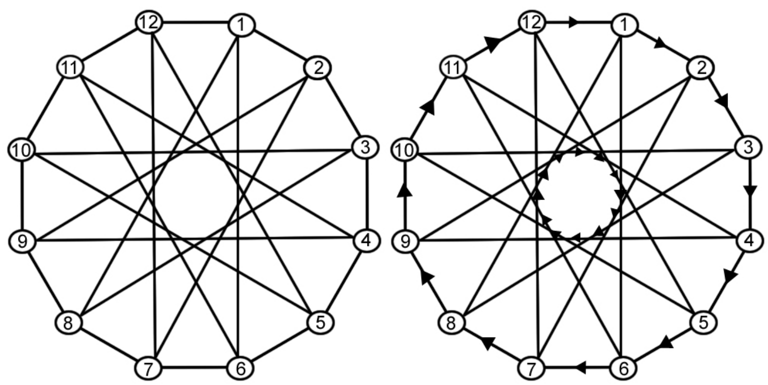

Figure: $C_{12}(1,5)$, and $C_{12}[1,5]$

Notice that a circulant graph will always be vertex-transitive by rotation. So when looking for a semi-transitive orientation for a circulant graph only edge and not dart transitivity need be established. Not every circulant digraph is going to be an orientation for the underlying graph. Below is a theorem that gives us a few cases when the circulant digraph is indeed a semi-transitive orientation.

Theorem 2.3 (Saner,Yuhasz,Wilson)**.** *If S is a subgroup of the multiplicative group U_N of units mod N and if S does not contain -1, then $\triangle = C_N[S]$ is a semi-transitive orientation for $\Gamma = C_N(S)$, and hence $C_N[S]$ is semi-transitive.*

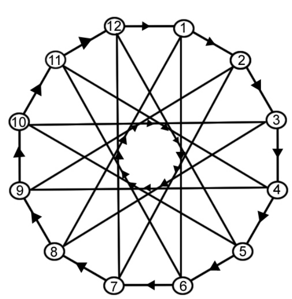

Figure: $C_{12}[1,5]$ is a semi-transitive orientation for $C_{12}(1,5)$ since $1, 5 \in U_N$

While this theorem gives us a lot of cases when the circulant digraph is a semi-transitive orientation, it still leaves a lot of cases open. The following theorem lets us expand the number of times the circulant digraph is indeed an orientation.

Proposition 2.4. *Let* $S = \{x|x \equiv 1 \pmod{n}$ *for* $2 < n < N$ *where* $n|N\}$. *Then* $C_N[S]$ *is a semi-transitive orientation for* $C_N(S)$.

Proof. Notice that $Aut(C_N[S])$ is transitive on vertices by rotation. I simply need to show that it is transitive on edges.

Let $(0, s), (0, t) \in E(C_N[S])$ be 2 edges coming from the vertex 0.

Consider the function μ where

$$\mu = (0)(0+n)(0+2n)...(0+N-n)(1, 1+n, 1+2n, ..., 1+N-n)...(n-1, n-1+n, ..., n-1+N-n)$$

That is, μ fixes every vertex of $C_N[S]$ that is equivalent to $0 \pmod{n}$, and μ sends every other vertex i to $i + n \pmod{n}$.

Let $(i, i + s_j)$ where $s_j \in S$ be a typical edge of $C_N[S]$. Notice that μ is a symmetry. If $i \equiv 0 \pmod{n}$, then $(i, i + s_j)\mu = (i, i + s_j + kn)$ for some $k \in \mathbb{N}$. Now $s_j + kn \equiv 1 + 0 \equiv 1 \pmod{n}$, so $s_j \in S$ and $(i, i + s_j)\mu \in E(C_N[S])$. If $i \not\equiv 0 \pmod{n}$, then $(i, i + s_j)\mu = (i + hn, i + hn + s_j)$ for some $h \in \mathbb{N}$, and since $s_j \in S$ then $(i, i + s_j)\mu \in E(C_N[S])$.Thus, μ is a symmetry of $Aut(C_N[S])$.

Finally, notice that since $s \equiv t \equiv 1 \pmod{n}$, μ will send $(0, s)$ to $(0, t)$, and $C_N[S]$ is a semi-transitive orientation for $C_N(S)$. □

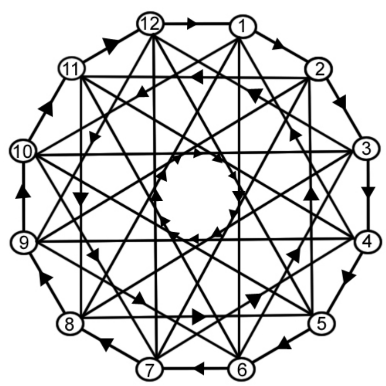

Figure: $C_{12}[1, 5, 9]$ is a semi-transitive orientation for $C_{12}(1, 3, 5)$ since $1, 5, 9 \equiv 1 \pmod{4}$

Notice that $\{1, 5, 9\}$ is not a subset of the multiplicative group of units mod 12 since 9 and 12 share a common divisor. This theorem allows us to expand the circulant digraphs we know to be semi-transitive orientations for the underlying circulant graph.

3. Consistent Cycles and Shunts

I will be concerned with the consistent cyclets and the shunts of the graph when trying to find a semi-transitive orientation for a graph. I believe these are key to find an orientation for the graph.

Definition 3.1. *In a graph* Γ, *a* cyclet *is a sequence* $[v_0, v_1, .., v_{r-1}]$ *of distinct vertices such that* $\{v_i, v_{i+1} \pmod{r}\} \in E(\Gamma)$ *for all* i.

Definition 3.2. *A* cycle *is a set consisting of a cyclet, all their shifts, and all their reversals.*

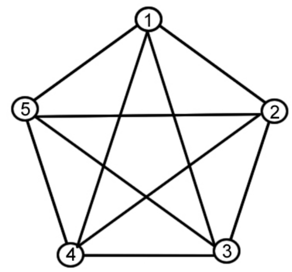

Figure: $[1, 2, 3, 4, 5]$ is a cyclet, $[2, 3, 4, 5, 1]$ is a shift, $[5, 4, 3, 2, 1]$ is a reversal

Notice that by the definition, a cyclet and a shift of the cyclet are different cyclets even though they are in the same cycle. This is a subtle distinction.

Definition 3.3. *A cyclet is* consistent *if there exists* $\sigma \in Aut(\Gamma)$ *such that* $v_i\sigma = v_{i+1} \pmod{r}$. *The symmetry* σ *that acts in this way is a* shunt.

That is, a cyclet is consistent if there exists a symmetry in the automorphism group of the graph such that the symmetry sends one vertex in the cyclet to the next one in the cyclet. This symmetry is a shunt.

The reader may be interested to know that there is a way to know how many different orbits of consistent cycles there are in a dart-transitive graph.

Theorem 3.4 (Biggs-Conway)**.** *If* Γ *is dart-transitive of degree* d *(*Γ *is regular since* Γ *is dart-transitive), then there are exactlly* $d-1$ *orbits of consistent cycles.*

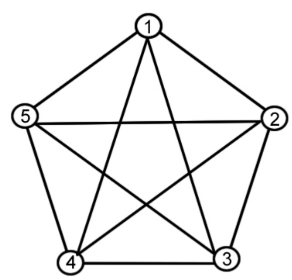

Figure: K_5 has 3 orbits of consistent cycles,
all triangles, all squares, and all pentagons

4. A Proposed Method of Finding a Semi-Transitive Orientation of Any Graph

Given any semi-transitive graph, how do we find a semi-transitive orientation of the graph? I propose a method to find the orientation and present many examples in the hopes to convince the reader that it works. The exact reason this method seems to work is open, and also a proof that this method always produces an orientation for a semi-transitive graph is open as well. I hope the reader can take the data I've accumulated and attempt their own proof of the result.

Let Γ be a semi-transitive graph. In order to find an orientation for it we will be concerned with all shunts from a vertex u to v. That is, pick an edge, and then look at all shunts that send the first vertex to the 2nd one. The number of shunts will be the same for each edge and will act in relatively the same way, so we need only consider one edge.

Take all shunts from u to v and consider all pairs of them. That is if $\alpha, \beta \in Aut(\Gamma)$ are 2 shunts from u to v, then check if $G =< \alpha, \beta >$ is vertex-transitive, edge-transitive, and not dart-transitive. If it is, then G will give a semi-transitive orientation for Γ.

In the following section I present the results after doing this search for a number of graphs. The reader should note how often this method works.

The first graph I will be concerned with is $J(24)$.

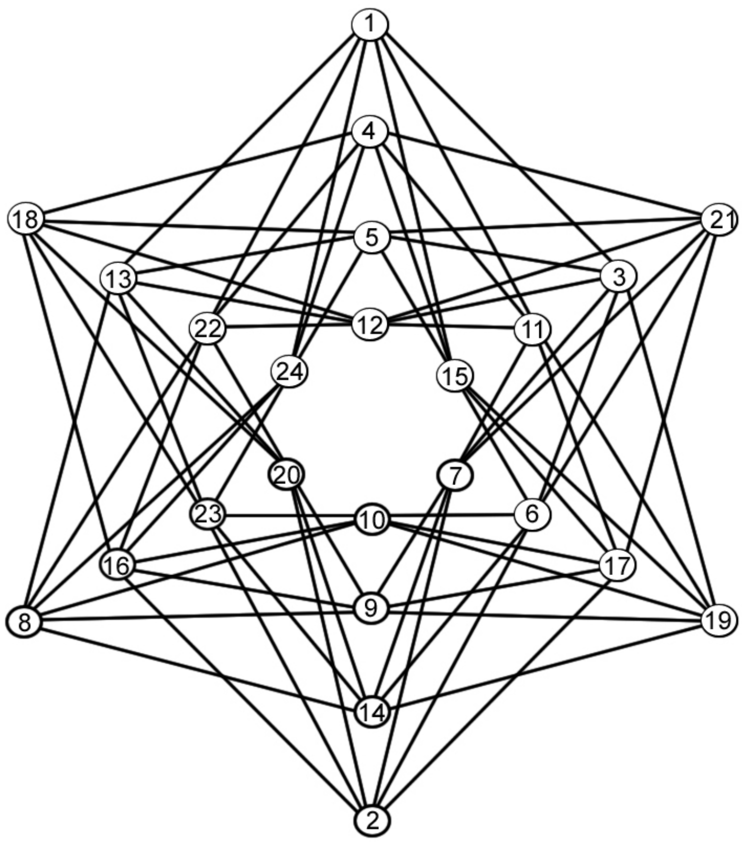

Figure: $J(24)$

J(24) has 24 vertices, 72 edges, and $Aut(J(24)) = 144$. That is a relatively small automorphism group for a graph. Because the automorphism group is small there are less shunts per edge. In this case I chose the edge $\{1, 3\}$ and checked all shunts that send the vertex 1 to the vertex 3. There are 10 shunts that do this.

$$
\begin{aligned}
a &= (1,3,6,10,8,13,5,15,19,14,23,24)(2,16,22,12,21,17,9,20,18,4,11,7)\\
b &= (1,3,6,2,16,22)(4,11,19,14,23,18)(5,21,17,9,8,13)(7,10,20,24,12,15)\\
c &= (1,3,5,15)(2,16,9,20)(4,11,12,21)(6,24,19,13)(7,18,14,22)(8,14,23,10)\\
d &= (1,3,5,21,4,11)(2,16,9,8,14,23)(6,18,17,22,19,13)(7,24)(10,20)(12,15)\\
e &= (1,3,19,14,8,13)(2,16,18,4,21,17)(5,11,6,9,23,22)(7,10,20,24,12,15)\\
f &= (1,3,19,14,8,13)(2,16,18,4,21,17)(5,15,6,10,23,24)(7,9,20,22,12,11)\\
g &= (1,3,7,9,8,13,12,11,19,14,20,22)(2,16,24,5,21,17,10,23,18,4,15,6)\\
h &= (1,3,7,2,16,24)(4,15,19,14,20,18)(5,11,6,9,23,22)(8,13,12,21,17,10)\\
i &= (1,3,12,11)(2,16,10,23)(4,15,5,21)(6,18,17,24)(7,22,19,13)(8,14,20,9)\\
j &= (1,3,12,21,4,15)(2,16,10,8,14,20)(5,11)(6,22)(7,18,17,24,19,13)(9,23)
\end{aligned}
$$

The following is a table that shows the shunts that successfully produced an orientation of the graph. The size of their automorphism group, the size of their alternets, and which orientation of the graph was produced is listed. I checked all 45 different pairs of shunts. There were 10 that produced a semi-transitive orientation for the graph. In this case, each pair of shunts produced exactly the same orientation. This was not the case in other examples I investigated. I was also able to determine that no 3 shunts would produce an orientation of $J(24)$.

α, β	$\|Aut\|$	$\|alt.\|$	STO
a,b	144	12	STO1
a,e	144	12	STO1
a,g	144	12	STO1
a,h	144	12	STO1
b,f	144	12	STO1
b,g	144	12	STO1
b,h	144	12	STO1
e,g	144	12	STO1
f,h	144	12	STO1
g,h	144	12	STO1

There are a few ways that shunts can fail to produce an orientation. The group generated by the shunts could either be too small in that it is not vertex-transitive or edge-transitive, or the group could be too big in that it is dart-transitive. In the following table I give a few pairs of shunts that fail to produce an orientation to illustrate the ways in which shunts could fail. The size of the group generated by the shunts, and the number of darts produced by the orientations is given for each pair that failed. If the number of darts produced was 72, then we would get a semi-transitive orientation since there would be one dart for each original edge of the graph. If the pair produces more than 72 then there are some edges that have more than 1 dart so it can not be an orientation.

α, β	$\lvert < \alpha, \beta > \rvert$	# darts
a,c	24	not V-T
c,d	48	not V-T
c,e	144	144
c,h	48	48
d,e	36	not V-T
d,f	144	144
d,g	144	144
e,f	24	not V-T
e,h	36	not V-T
f,g	24	not V-T
f,i	144	144
g,i	24	not V-T
h,j	36	not V-T
i,j	48	not V-T

Here is the orientation produced:

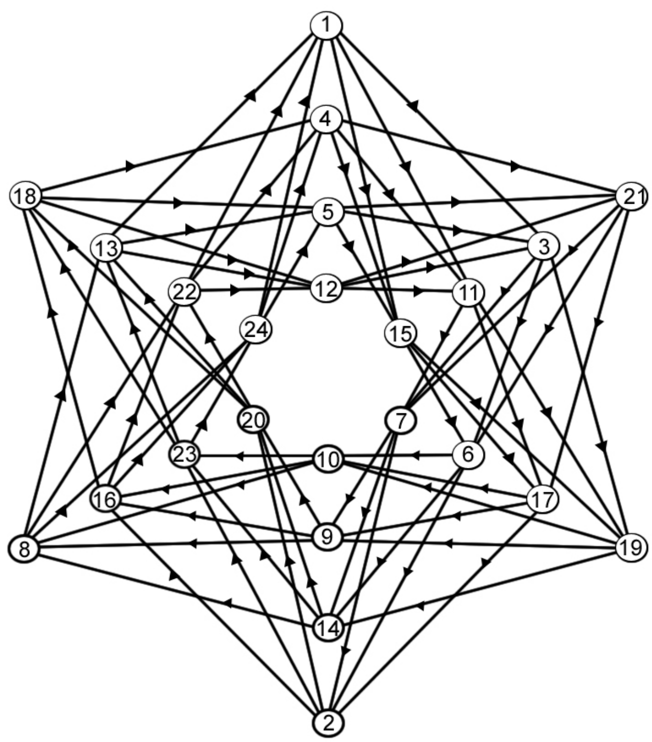

Figure: $J(24)$ STO1

The next graph I investigated was $R_8(2,3)$.

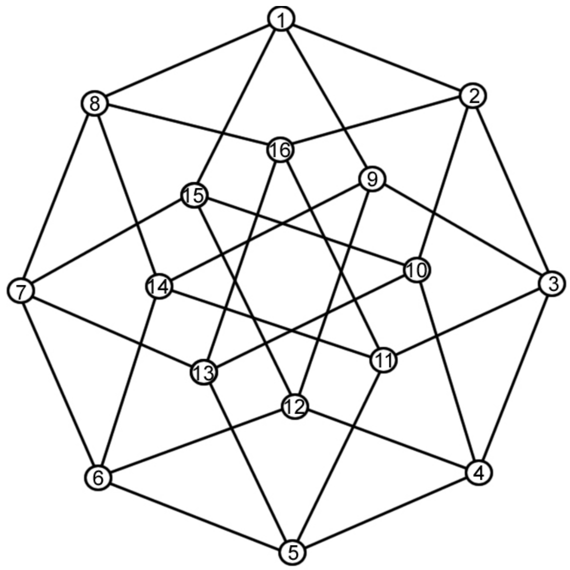

Figure: $R_8(2,3)$

$R_8(2,3)$ has 16 vertices, 32 edges, and $Aut(R_8(2,3)) = 384$. In this case I chose the edge $\{1,2\}$ and checked almost all shunts that send the vertex 1 to the vertex 2. There are 18 shunts that do this, so I was unable to check every pari of shunts. $R_8(2,3)$ is known to have 2 non-isomorphic orientations which I was able to find even after not checking every pari of shunts.

$$\begin{array}{rcl}
a &=& (1,2,10,13,7,8)(3,4,5,6,14,9)(11,12)(15,16) \\
b &=& (1,2,10,15)(3,4,12,9)(5,6,14,11)(7,8,16,13) \\
c &=& (1,2,10,13,5,6,14,9)(3,15,16,4,7,11,12,8) \\
d &=& (1,2,10,4,12,9)(3,15)(5,6,14,18,16,13)(7,11) \\
e &=& (1,2,10,4,5,6,14,8)(3,13,12,11,7,9,16,15) \\
f &=& (1,2,10,15)(3,13,12,8)(4,7,9,16)(5,6,14,11) \\
g &=& (1,2,16,13,7,15)(3,11,5,6,12,9)(4,14)(8,10) \\
h &=& (1,2,16,8)(3,11,14,9)(4,5,6,12)(7,15,10,13) \\
i &=& (1,2,16,11,5,6,12,15)(3,13,14,4,7,9,10,8) \\
j &=& (1,2,16,8)(3,13,14,15)(4,5,6,12)(7,9,10,11) \\
k &=& (1,2,16,13,5,6,12,9)(3,8,10,11,7,4,14,15) \\
l &=& (1,2,16,11,14,9)(3,8)(4,7)(5,6,12,15,10,13) \\
m &=& (1,2,3,11,5,6,7,15)(4,14,13,12,8,10,9,16) \\
n &=& (1,2,3,11,14,8)(4,5,6,7,15,10)(9,16)(12,13) \\
o &=& (1,2,3,4,12,15)(5,6,7,8,16,11)(9,10)(13,14) \\
p &=& (1,2,3,4,5,6,7,8)(9,10,11,12,13,14,15,16) \\
q &=& (1,2,3,9)(4,14,15,16)(5,6,7,13)(8,10,11,12) \\
r &=& (1,2,3,9)(4,12,15,10)(5,6,7,13)(8,16,11,14)
\end{array}$$

The following is a table that shows the shunts that successfully produced an orientation of the graph. The size of their automorphism group, the size of their alternets, and which orientation of the graph was produced is listed. Even though I did not check every pair of shunts I was able to find 9 that produced one orientation up to isomorphism, and 1 that produced a different non-isomorphic orientation.

s,t	$\|Aut\|$	$\|alt.\|$	STO
c,f	64	4	STO1
p,q	64	4	STO2≅1
k,m	64	4	STO2≅1
j,f	64	4	STO3≅1
q,f	64	4	STO2≅1
e,f	64	4	STO3≅1
f,i	64	4	STO3≅1
f,m	64	4	STO1
c,m	64	4	STO1
b,k	32	8	STO4≇1

Here are the orientations produced:

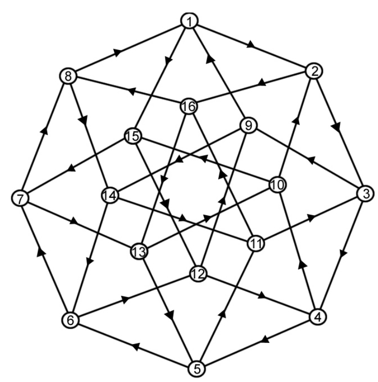

Figure: $R_8(2,3)STO2$

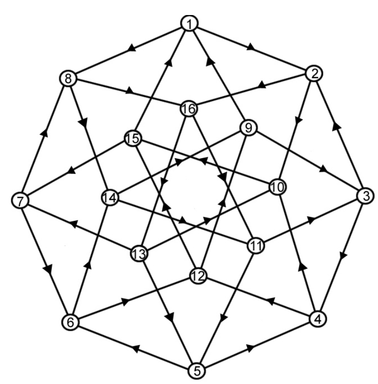

Figure: $R_8(2,3)STO4$

The next graph I investigated was $PS(3,7;2)$.

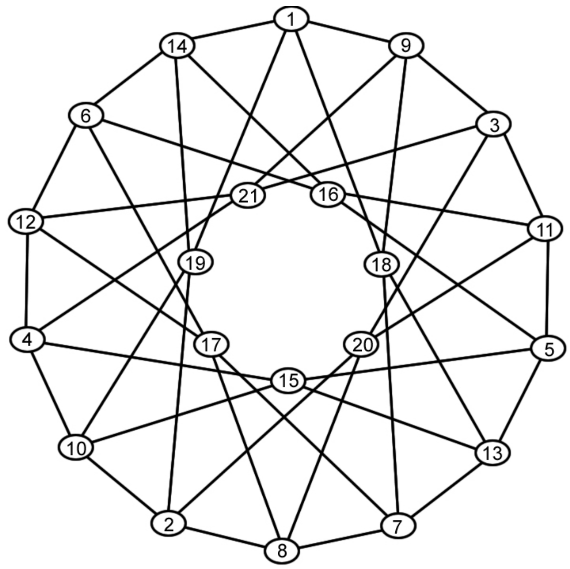

Figure: $PS(3,7;2)$

$PS(3,7;2)$ has 21 vertices, 42 edges, and $Aut(PS(3,7;2)) = 336$. In this case I chose the edge $\{1,9\}$ and checked all shunts that send the vertex 1 to the vertex 9. There are 12 shunts that do this.

$$
\begin{aligned}
a &= (1,9,18)(2,20,8)(3,7,19)(4,5,6)(10,11,17)(12,15,16)(13,14,21)\\
b &= (1,9,18)(2,12,5)(3,7,14)(4,15,10)(6,11,8)(13,19,21)(16,20,17)\\
c &= (1,9,18)(2,11,15)(3,13,19)(4,8,16)(5,10,20)(6,12,17)(7,14,21)\\
d &= (1,9,18)(2,4,17)(3,13,14)(5,16,11)(6,20,15)(7,19,21)(8,10,12)\\
e &= (1,9,3,20,2,19)(4,16,7)(5,17,15,6,13,12)(8,10,14,18,21,11)\\
f &= (1,9,3,20,8,17,6,14)(2,7,12,16,19,18,21,11)(4,5,10,13)\\
g &= (1,9,3,11,16,14)(2,13,12)(4,8,15,17,10,7)(5,6,19,18,21,20)\\
h &= (1,9,3,11,5,15,10,19)(2,14,18,21,20,16,13,4)(6,7,12,8)\\
i &= (1,9,21,4,10,19)(2,14,18,3,12,15)(5,8,16,7,11,17)(6,13,20)\\
j &= (1,9,21,4,15,5,16,14)(2,7,20,17)(3,12,10,13,11,6,19,18)\\
k &= (1,9,21,12,17,8,2,19)(3,4,6,7,20,10,14,18)(11,15,16,13)\\
l &= (1,9,21,12,6,14)(2,13,20,15,8,5)(3,4,17,16,19,18)(7,11,10)
\end{aligned}
$$

The following is a table that shows the shunts that successfully produced an orientation of the graph. The size of their automorphism group, the size of their alternets, and which orientation of the graph was produced is listed. Even though I did not check every pair of shunts I was able to find 4 that produced one orientation up to isomorphism, and 4 that produced a different non-isomorphic orientation.

s,t	$\lvert Aut\rvert$	$\lvert alt.\rvert$	STO
a,g	42	14	STO1
a,i	42	14	STO2≇1
b,e	42	14	STO2≇1
b,l	42	14	STO1
c,g	42	14	STO3≅2
c,i	42	14	STO4≅1
d,e	42	14	STO4≅1
d,l	42	14	STO3≅2

5. The A Set

What characteristics of 2 shunts produces a semi-transitive orientation? By looking at the similarities in the 2 shunts that produce an orientation I noticed that these shunts share something in common, the A-set.

Definition 5.1. *The* A-set *is the set* $A = \{a \in V(\Gamma) | a\alpha = a\beta\}$ *for the 2 shunts* $\alpha, \beta \in Aut(\Gamma)$.

A vertex is in the A-set if both shunts send that vertex to the same place. The converse of the A-set is the B-set, the set of vertices that the vertices in the A-set get sent to.

Definition 5.2. *The* B-set *is the set* $B = \{b \in V(\Gamma) | b\boldsymbol{s}^{-1} = b\boldsymbol{t}^{-1}\}$ *for the 2 shunts* $\alpha, \beta \in Aut(\Gamma)$.

Example 5.3. *Take the 2 shunts* a, e *from* $J(24)$ *that I analyzed above.*

$$a = (1, 3, 6, 10, 8, 13, 5, 15, 19, 14, 23, 24)(2, 16, 22, 12, 21, 17, 9, 20, 18, 4, 11, 7)$$

$$e = (1, 3, 19, 14, 8, 13)(2, 16, 18, 4, 21, 17)(5, 11, 6, 9, 23, 22)(7, 10, 20, 24, 12, 15)$$

So the A and B sets are:

$$A = \{1, 2, 8, 18, 19, 21\}$$

$$B = \{3, 16, 13, 4, 14, 17\}$$

I noticed that the A and B sets for 2 shunts that produced an orientation were usually much larger than those of shunts that do not produce an semi-transitive orientation of the graph.

Below is the A-sets for all shunts that produced a semi-transitive orientation from the graph $J(24)$ that I analyzed in the set above.

s,f	$A = \{a \in V(\Gamma) \| a\mathbf{s}= a\mathbf{t}\}$	$\|A\|$
a,b	1,2,8,18,19,21,3,4,14,13,16,17	12
g,b	1,2,8,18,19,21,5,6,9,11,22,23	12
a,e	1,2,8,18,19,21	6
g,e	1,2,8,18,19,21	6
a,h	1,2,8,18,19,21,7,10,12,15,20,24	12
g,h	1,2,8,18,19,21,3,4,13,14,16,17	12
a,g	1,2,8,18,19,21	6
b,f	1,2,8,18,19,21	6
b,h	1,2,8,18,19,21	6
f,h	1,2,8,18,19,21	6

You will notice that some similarities exist between the A-sets of all the set. Here is the A-set for all the shunts that produced a semi-transitive orientation from the the graph $R_8(2,3)$ that I analyzed above.

s,f	$A = \{a \in V(\Gamma) \| a\mathbf{s}= a\mathbf{t}\}$	$\|A\|$
c,f	1,5,12,16,2,4,6,8	8
m,f	1,5,12,16,9,11,13,15	8
e,f	1,5,3,7,2,6,9,13	8
i,f	1,5,3,7,4,8,11,15	8
j,f	1,5,3,7	4
q,f	1,5,12,16	4
p,q	1,5,2,6,1,14,11,15	8
c,m	1,5,12,16	4
k,m	1,5,4,8	4
b,k	1,5,12,16	4

6. Open Questions

The exact reason that shunts seem to produce semi-transitive orientations rather quickly still remains a question. I present 2 conjectures that I believe to be true. Then I leave the reader with the challenge of proving these conjectures using all the information provided.

Conjecture 6.1. *Let $G =< \alpha, beta >$ be vertex-transitive where α, β are shunts from u to v in the degree 4 graph Γ. Let $A = \{a \in V(\Gamma) | a\boldsymbol{s}= a\boldsymbol{t}\}$. Let $B = \{b \in V(\Gamma) | b\boldsymbol{s}^{-1} = b\boldsymbol{t}^{-1}\}$. If $u\boldsymbol{s}^{-1} = n_1$ and $u\boldsymbol{t}^{-1} = n_2$ where $n_1 \neq n_2$ and the set of neighbors of u, $\Gamma(u) \subseteq A \cup B \cup \{n_1, n_2\}$, then G is edge-transitive and not dart-transitive. Thus, we get a semi-transitive orientation for the graph Γ.*

I believe this is true and there is a partial proof of it available. The next conjecture I also believe to be true. But it is a much more wide reaching theorem. I suspect that a proof of it would be very involved. I invite the reader to try and solve both conjectures.

Conjecture 6.2. *Every Semi-Transitive Orientation of a Semi-Transitive Graph Γ can be found using $G =< \alpha, \beta >$ where α, β are shunts from u to v in Γ.*

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