NORTHERN ARIZONA UNIVERSITY

Determining Resonance Varieties of Hyperplane Arrangements

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Abstract

This document discusses the research conducted by the author for the NAU Math REU program during the summer of 2008. First, we review the necessary algebraic background for working with hyperplane arrangements. Then we discuss the Orlik-Solomon algebra and matroids of arrangements, and build up the concept of neighborly partitions. This paper discusses the efforts made in working with neighborly partitions, and their use in determining the resonance varieties of an arrangement. Also included are several packages written in Mathematica, which assisted in the computations used in working with examples.

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1 Introduction

This document details the focus and results of my NSF-granted research at Northern Arizona University in the summer of 2008, under Professor Michael J. Falk. The topic of the program was *Hyperplane Arrangements*. My area of focus was determining resonance varieties of hyperplane arrangements. The resonance varieties give a strong invariant of the structure of the Orlik-Solomon algebra, and can be useful for classifying the topology of the complements of hyperplane arrangements. In this document I give an overview of the background research conducted in the initial weeks; discuss how to represent hyperplane arrangements using different diagrams, including the matroid respresentation; demonstrate how neighborly partitions are created from the construction of nested sets, and finally explain how all these topics tie in to determining the degree-1 resonance varieties of an arrangement. A small part of the research period was devoted to learning how to program in Mathematica in order to write packages for assisting in hyperplane arrangement calculations.

2 Background

The majority of the first three weeks of the program were devoted to a comprehensive introduction to the topic of Hyperplane Arrangements. Professor Falk met with my co-researcher Nakul Dawra and I on a daily basis to go over concepts. In addition to our meetings, my task for the first few weeks was to develop a solid understanding of the algebraic structures and concepts necessary to be able to follow the topics being introduced. The result of the initial research period was my write-up of a document of useful definitions along with important ideas and theorems. I prepared a similar document in the sixth week with more algebra concepts. In the following subsections, I describe the concepts necessary to develop a mathematical foundation for my research topic.

2.1 Arrangements of Hyperplanes

We start by giving a formal definition of a hyperplane and hyperplane arrangement, and then define the terms that will be necessary for understanding the rest of the paper.

Definition 2.1. Let $\mathbb{K}$ be a field and let $V_{\mathbb{K}}$ be a vector space of dimension l. A **hyperplane** H in $V_{\mathbb{K}}$ is an affine subspace of dimension $(l-1)$. A **hyperplane arrangement** $\mathcal{A}_{\mathbb{K}} = \{H_1, \ldots, H_n\}$ is a finite set of hyperplanes in $V_{\mathbb{K}}$.

Definition 2.2. Let V be a vector space. The **dual space** V^* of V is the set of all linear functionals (linear maps from V to the field underlying V) on V, i.e., scalar-valued linear maps on V.

Note. A *subspace arrangement* is a finite set of affine subspaces of V with no dimension restrictions. We call $\mathcal{A}$ an *l-arrangement* when we want to emphasize the dimension of V.

Definition 2.3. $\mathcal{A}$ is a **complexified (real) arrangment** iff all H_i's are defined by real equations.

Definition 2.4. The product

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called the **defining polynomial** of $\mathcal{A}$. For example, the 2-arrangement consisting of the lines $y = 0$, $x = 0$, and $y = x$ in $\mathbb{R}^2$ is denoted as

$$Q(\mathcal{A}) = xy(x - y).$$

For another example, the central 3-arrangement defined by

$$Q(\mathcal{A}) = xyz(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$$

describes the nine planes of symmetry of the cube with vertices at $(\pm 1, \pm 1, \pm 1)$. This arrangement is typically referred to as the **B_3-arrangement**.

Definition 2.5. We call $\mathcal{A}$ **centerless** if the union of the hyperplanes in $\mathcal{A}$ have an empty intersection. If there is a nonempty intersection, we call $\mathcal{A}$ **centered** with **center** T. When we want to emphasize that an arrangement can be either centered or centerless we call it **affine**.

Definition 2.6. Let $\mathbf{L}(\mathcal{A})$ be the set of all nonempty intersections of elements in $\mathcal{A}$. We agree that $L(\mathcal{A})$ includes V as the intersection of the empty collection of hyperplanes.

Definition 2.7. Let $\mathcal{A}$ be an arrangement on V. If $\mathcal{B} \subseteq \mathcal{A}$ is a subset, then $\mathcal{B}$ is called a **subarrangement**. For $X \in L(\mathcal{A})$ define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

In other words, $\mathcal{A}_X$ is the set of all hyperplanes that contain X. Note that $\mathcal{A}_V$ is the empty arrangement and if $X \neq V$, then $\mathcal{A}_X$ has center X in any arrangement. Define an arrangement $\mathcal{A}^X$ on X by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

We call $\mathcal{A}^X$ the **restriction** of $\mathcal{A}$ to X. Note that $\mathcal{A}^V = \mathcal{A}$.

2.2 The Set $L(\mathcal{A})$

Note. For an arrangement $\mathcal{A}$ and set of nonempty intersections $L = L(\mathcal{A})$, we define the partial order on L by

$$X \leq Y \Leftrightarrow Y \subseteq X.$$

This is reverse inclusion. Thus V is the unique minimal element. For central arrangements, L has a unique maximal element.

Definition 2.8. Define a **rank function** on $L(\mathcal{A})$ by $r(X) = \text{codim}X$. Thus $r(V) = 0$ and $r(H) = 1$ for $H \in \mathcal{A}$. Call H an **atom** of $L(\mathcal{A})$.

Definition 2.9. Let $X, Y \in L$. Define their **meet** by $X \wedge Y = \cap \{Z \in L \mid X \cup Y \subseteq Z\}$. If $X \cap Y \neq \emptyset$, we define their **join** by $X \vee Y = X \cap Y$.

Definition 2.10. The **rank** of $\mathcal{A}$, $r(\mathcal{A})$, is the rank of a maximal element of $L(\mathcal{A})$. Call the l-arrangement $\mathcal{A}$ **essential** if $r(\mathcal{A}) = l$, If $\mathcal{A}$ is a central arrangement, the center is the **unique maximal element** of $L(\mathcal{A})$.

Lemma 2.1. *Maximal elements of $L(\mathcal{A})$ have the same rank. This is clear if $\mathcal{A}$ is a central arrangement, since $L(\mathcal{A})$ has a unique maximal element. For the proof of a centerless arrangement, see Orlik and Terao, Lemma 2.4.*

Definition 2.11. Call the arrangements $\mathcal{A}$ over V and $\mathcal{B}$ over W **lattice equivalent**, or L-equivalent, if there is an order preserving bijection $\pi : L(\mathcal{A}) \to L(\mathcal{B})$.

Additionally, we will define the term *k-generic* in the context of hyperplane arrangements. This property of an arrangement gives us good insight into the structure of the intersection lattice:

Definition 2.12. An arrangement $\mathcal{A}$ in $V_{\mathbb{K}}$ is **k-generic** if its intersection lattice agrees the Boolean arrangement in $V_{\mathbb{K}}$ up to rank k.

2.3 Graphic Arrangements

Definition 2.13. Let $\mathbb{K}$ be a field and let $V = \mathbb{K}^l$. Let $x_1, \ldots, x_l$ be a basis for the dual space V^*. Given the graph $G = (\mathcal{V}, \mathcal{E})$ with l vertices, define an arrangement $\mathcal{A}(G)$ by

$$\mathcal{A}(G) = \{\ker(x_i - x_j) \mid \{i, j\} \in \mathcal{E}\}.$$

The arrangement $\mathcal{A}(G)$ is called a **graphic arrangement**.

Example 2.1. Let $\mathbb{K} = \mathbb{R}$. For the complete graph with l vertices, $\mathcal{A}(G)$ has a defining polynomial

$$Q(\mathcal{A}(G)) = \prod_{1 \leq i < j \leq l)} (x_i - x_j).$$

Note that the edge ij contributes $(x_i - x_j)$ to the product.

2.4 Matroid Theory

A characteristic of matroids is that they can be defined in many different but equivalent ways. Matroids prove to be a very useful tool in working with hyperplane arrangements.

Definition 2.14. A **matroid** M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:

(1) $\emptyset \in \mathcal{I}$.

(2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Note. If M is the matroid $(E, \mathcal{I})$, then M is called a matroid on E. The members of $\mathcal{I}$ are the **independent sets** of M, and E is the **ground set** of M. A subset of E that is not in $\mathcal{I}$ is called **dependent**.

Definition 2.15. A minimal dependent set in an arbitrary matroid M (dependent sets all of whose proper subsets are independent), is called a **circuit** of M. The set of circuits of M is denoted by $\mathcal{C}$ or $\mathcal{C}(M)$. A circuit having n elements is called a n**-circuit**.

Note. For any set of circuits $\mathcal{C}$, the empty set is not in $\mathcal{C}$ and if C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

Lemma 2.2. *If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \subseteq C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$. For the proof see Oxley, Lemma 1.1.3.*

Proposition 2.3. *Let E be the set of edges of a graph G and $\mathcal{C}$ be the set of edge sets of cycles of G. Then $\mathcal{C}$ is the set of circuits of a matroid on E. For the proof, see Oxley, Proposition 1.1.7.*

2.5 The Möbius Function

Definition 2.16. Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. Define the **Möbius function** $\mu_{\mathcal{A}} = \mu : L \times L \to \mathbb{Z}$ as follows:

$$\begin{array}{rl} \mu(X,X) = 1 & \text{if } X \in L, \\ \sum_{X \leq Z \leq Y} \mu(X,Z) = 0 & \text{if } X,Y,Z \in L \text{ and } X < Z, \\ \mu(X,Y) = 0 & \text{otherwise.} \end{array}$$

Proposition 2.4 (Möbius inversion formula)**.** *Let f, g be functions on $L(\mathcal{A})$ with values in an abelian group. Then*

$$\begin{aligned} g(Y) &= \sum_{X \in L_Y} f(X) \Leftrightarrow f(Y) = \sum_{X \in L_Y} \mu(X,Y) g(X) \\ g(X) &= \sum_{Y \in L^Y} f(Y) \Leftrightarrow f(X) = \sum_{X \in L^X} \mu(X,Y) g(Y). \end{aligned}$$

Theorem 2.5. *For an arrangement $\mathcal{A}$,*

$$\dim \mathcal{A}^p = \sum_{r(X)=p} (-1)^{p-1} \mu(X).$$

Stated on June 3, 2008.

2.6 The Poincaré Polynomial

Definition 2.17. Let $\mathcal{A}$ be an arrangement with intersection poset L and Möbius function μ. Let t be an indeterminate. Define the **Poincaré Polynomial** of $\mathcal{A}$ by

$$\pi(\mathcal{A},t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}.$$

$\pi(\mathcal{A},t)$ always has nonnegative coefficients. If $\mathcal{A}$ is empty, then $\pi(\mathcal{A},t) = 1$.

Lemma 2.6. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrangements and let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. Then*

$$\pi(\mathcal{A},t) = \pi(\mathcal{A}_1,t)\pi(\mathcal{A}_2,t).$$

The proof can be found in Orlik and Terao, Lemma 2.50.

Definition 2.18. Define the **characteristic polynomial** of $\mathcal{A}$ by

$$\chi(\mathcal{A},t) = t^l \pi(\mathcal{A},-t^{-1}) = \sum_{X \in L} \mu(X) t^{\dim(X)}.$$

Note that $\chi(\mathcal{A},t)$ is a polynomial of degree l with a leading coefficient of 1.

Note. If we can draw the Hasse diagram of an arrangement $\mathcal{A}$ (a lattice with the vector space as the lowest node, and the center (if applicable) at the top, with all the intersections in $L(\mathcal{A})$ drawn according to the ordering of reverse inclusion), we can easily determine the characteristic polynomial. If we set the lowest node to 1, then for each increasing level, assign each node a number determined by the negative sum of all children nodes. The level, or rank, determines the degree of the node's contribution. Thus the lowest level contributes a 1, the next level contributes (negative) linear terms, then next level contributes quadratic terms, the next level (negative) cubic terms, and so forth.

Example 2.2. The arrangement defined by the polynomial

$$Q(\mathcal{A}) = xy(x+y)$$

has a characteristic polynomial computed as follows: The vector space V contributes a 1. For the next level, we have the lines $x = 0$, $y = 0$, and $x + y = 0$, each marked -1. The top level, the intersection defined by the point $(0,0)$, is the negative of the sum $1 - 1 - 1 - 1$, or 2. Thus the characteristic polynomial is $1 + 3t + 2t^2$.

2.7 Algebra Review

Definition 2.19. Let R be a ring (not necessarily commutative nor with 1). A **left** R**-module** or a **left-module over** R is a set M together with

(1) a binary operation $+$ on M under which M is an abelian group, and

(2) an action of R on M (that is, a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies

   (a) $(r+s)m = rm + sm, \quad$ for all $r, s \in R$, $m \in M$,

   (b) $(rs)m = r(sm), \quad$ for all $r, s \in R$, $m \in M$, and

   (c) $r(m+n) = rm + rn, \quad$ for all $r \in R$, $m, n \in M$.

   If the ring R has a 1 we impose the additional axiom:

   (d) $1m = m, \quad$ for all $m \in M$.

If the ring R is commutative, we can make a left R-module into a right R-module.

Note. When R is a field F, the axioms for an R-module are precisely the same as those for a vector space over F.

Example 2.3. Any abelian group A can be made into a $\mathbb{Z}$-module.

Definition 2.20. Let R be a ring and let M and N be R-modules. Define $\mathbf{Hom}_R(M, N)$ to be the set of all R-module homomorphisms from M into N.

Definition 2.21. Let A, B, and C be R-modules over some ring R. Consider a pair of homomorphisms

$$A \xrightarrow{\psi} B \xrightarrow{\phi} C$$

with image $\psi = \ker \phi$. This pair of homomorphisms is said to be **exact** (at B).

Definition 2.22. A sequence

$$\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$$

of homomorphisms is said to be an **exact sequence** if it is exact at every X_n between a pair of homomorphisms.

Proposition 2.7. *Let A, B, and C be R-modules over some ring R. Then*

(1) *The sequence $0 \to A \xrightarrow{\psi} B$ is exact (at A) if and only if ψ is injective.*

(2) *The sequence $B \xrightarrow{\psi} C \to 0$ is exact (at C) if and only if ϕ is surjective.*

Corollary 2.8. *The sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$ is exact if and only if ψ is injective, ϕ is surjective, and image $\psi = \ker \phi$, i.e., B is an extension of C by A.*

Definition 2.23. The exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$ is called a **short exact sequence**.

Definition 2.24. Let $\mathcal{C}$ be a sequence of abelian group homomorphisms:

$$0 \to C^0 \xrightarrow{d_1} C^1 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} C^{n-1} \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} \cdots .$$

(1) The sequence $\mathcal{C}$ is called a **cochain complex** if the composition of any two successive maps is zero: $d_{n+1} \circ d_n = 0$ for all n.

(2) If $\mathcal{C}$ is a cochain complex, its $n^{\mathbf{th}}$ **cohomology group** is the quotient group $\ker d_{n+1}/\text{image} d_n$, and is denoted by $H^n(\mathcal{C})$.

Theorem 2.9 (The Long Exact Sequence in Cohomology)**.** *Let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes. Then there is a long exact sequence of cohomology groups:*

$$0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \xrightarrow{\delta_0} H^1(\mathcal{A})$$
$$\to H^1(\mathcal{B}) \to H^1(\mathcal{C}) \xrightarrow{\delta_1} H^2(\mathcal{A}) \to \cdots$$

where the maps between cohomology groups at each level are those in the previous proposition. The maps δ_n are called ***connecting homomorphisms****.*

Definition 2.25. A **graded ring** A is a ring that has a direct sum decomposition into (abelian) additive groups

$$A = \bigoplus_{n \in \mathbb{N}} A_n$$

such that $A_s A_r \subseteq A_{s+r}$.

Definition 2.26. A **graded module** M over a graded ring A is defined as

$$M = \bigoplus_{i \in \mathbb{N}} M_i$$

such that $A_i M_j \subseteq M_{i+j}$.

Definition 2.27. A **graded algebra** over a graded ring A is an A-algebra E which is both a graded A-module and a graded ring. Thus

$$E = \bigoplus_i E_i$$

such that $A_i E_i \subset E_{i+j}$ and $E_i E_j \subset E_{i+j}$.

3 The Orlik-Solomon Algebra

The *Orlik-Solomon algebra* an important structure that is used frequently in the study of hyperplane arrangements. The rest of this paper will make use of it, particularly in explaining the importance of neighborly partitions.

3.1 The Exterior Algebra

Before we can create the Orlik-Solomon algebra (hereafter referred to as the *O-S algebra*), we must first construct a structure called the *exterior algebra.*

Definition 3.1. Let $\mathcal{A}$ be an arrangement over a field $\mathbb{K}$, and let $\mathcal{K}$ be a commutative ring. Let

$$E_1 = \bigoplus_{H \in \mathcal{A}} \mathcal{K} e_H$$

and let $E = E(\mathcal{A}) = \Lambda(E_1)$ be the **exterior algebra** of E_1. $e_H^2 = 0$ and $e_H e_K = -e_K e_H$ for all $H, K \in \mathcal{A}$. If $|\mathcal{A}| = n$, then

$$E = \bigoplus_{p=0}^{n} E_p,$$

where $E_0 = \mathcal{K}$, E_1 is as defined above, and E_p consists of the product $e_{H_1} \cdots e_{H_p}$ with $H_k \in \mathcal{A}$.

Additionally, we define a boundary mapping $\partial : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1$ and for $p \geq 2$

$$\partial(e_{H_1} \cdots e_{H_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \cdots e_{H_{k-1}} \cdot e_{H_{k+1}} \cdots e_{H_p}$$

for all $H_1, \ldots, H_p \in \mathcal{A}$.

3.2 Dependent Sets

One final concept is necessary for describing the O-S algebra, the notion of dependence. Given a p-tuple of hyperplanes, $S = (H_1, \ldots, H_p)$, write $|S| = p$, $e_S = e_{H_1} \cdots e_{H_p} \in E$, and $\cap S = H_1 \cap \cdots \cap H_p$.

Definition 3.2. The p-tuple S is **independent** if the corresponding linear forms $\alpha_1, \ldots, \alpha_p$ are linearly independent.

The property is more easily explained through example.

Example 3.1. For the hyperplane arrangement shown in Figure 1, the defining equations are $x = 0$, $y = 0$ and $x = -y$. Thus the matrix of the linear forms is

$$\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The set $S = (x = 0, x = -y)$ is an example of an independent 2-tuple.

3.3 The Orlik-Solomon Algebra A

Let $\mathcal{A}$ be an arrangement. Let $I = I(\mathcal{A})$ be the ideal of E generated by ∂e_s for all dependent $S \in \mathbf{S}$. For example, if the set $S = (H_1, H_2, H_3)$ is dependent, then $\partial(e_1 e_2 e_3) = e_{12} - e_{13} + e_{23} \in I$.

Definition 3.3. Let $\mathcal{A}$ be an arrangement. The **Orlik-Solomon algebra** $A = A(\mathcal{A})$ is the quotient group of the exterior algebra E and the ideal I, i.e., $A = E/I$.

The O-S algebra defines the structure and relations that are necessary for understanding resonance varieties.

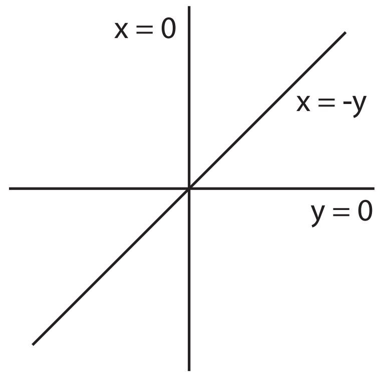

Figure 1: A simple arrangement containing three hyperplanes

4 Matroids of Arrangements

One of the most useful visual depictions of hyperplane arrangements is the matroid. In this section we discuss how matroids are constructed through the example of the braid arrangement in three dimensions.

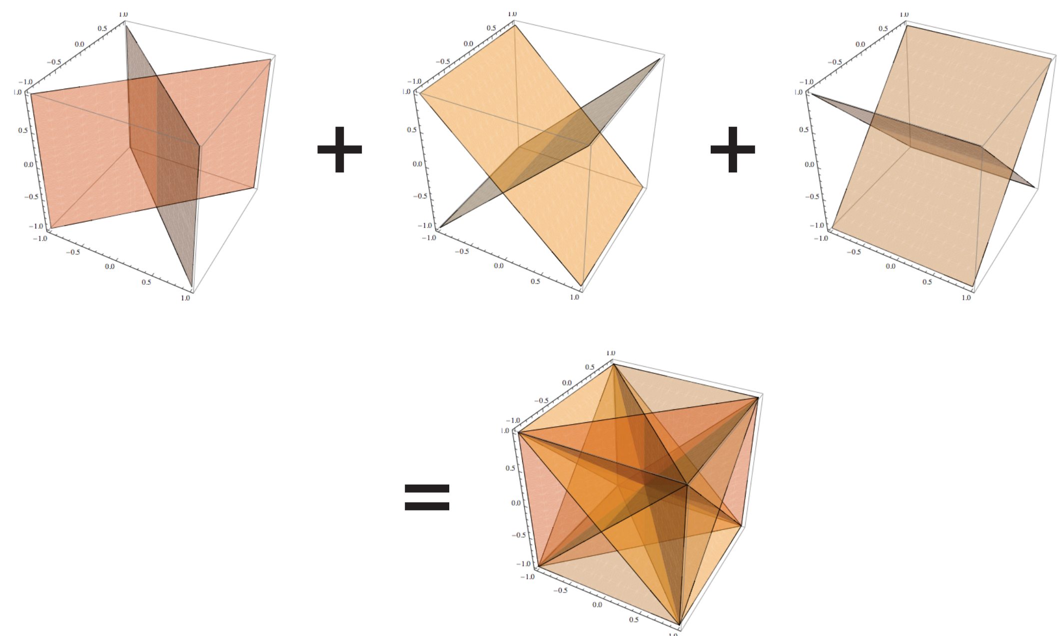

Figure 2: The construction of the Braid Arrangement in 3 dimensions

4.1 Projective Diagrams

The Braid 3-arrangement consists of taking the intersection of the six planes corresponding to the defining equations $x = y$, $x = -y$, $x = z$, $x = -z$, $y = z$, and $y = -z$ (see Figure 2).We can draw a *projective diagram* by viewing the lines formed by intersecting the arrangement with the plane $z = 1$ and giving each hyperplane a number, and each intersection a set of numbers, as in Figure 3. Note that each set of parallel lines is understood to intersect "at infinity" as they do not actually intersect in the plane $z = 1$.

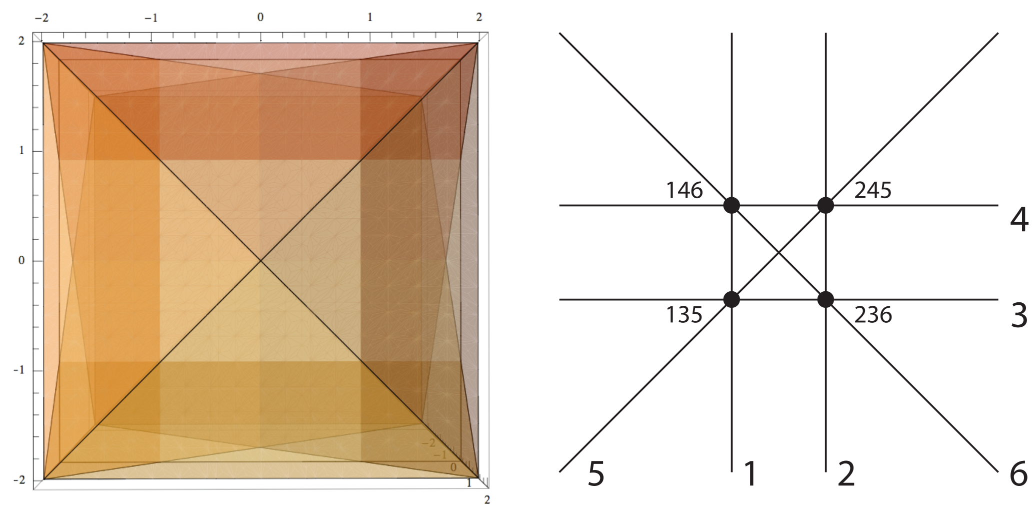

Figure 3: Projective Diagram (*right*) created by intersecting the Braid 3-Arrangement with the plane $z = 1$ *(left)*

4.2 Matroids

We can then construct a Hasse diagram, which, to remind the reader, is a lattice with the vector space as the lowest node, and the center (if applicable) at the top, with all the intersections in $L(\mathcal{A})$ drawn according to the ordering of reverse inclusion (see Figure 4). Note that in Figure 4, the intersecting pairs of hyperplanes are omitted for the sake of readability.

The matroid, if realizable, is drawn such that any set of three or more collinear points in the matroid signifies that they form a flat (intersection of hyperplanes) of rank 2, and any set of coplanar points represents a rank-3

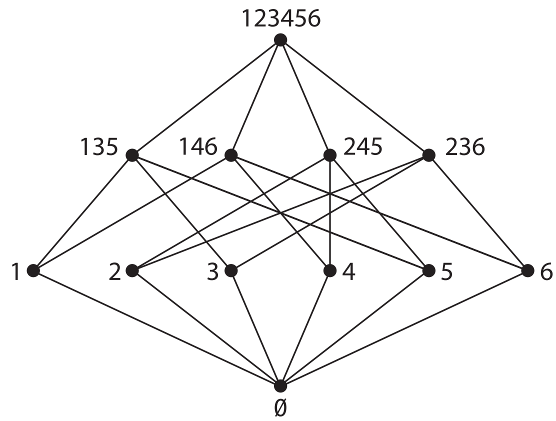

Figure 4: Hasse diagram of the Braid 3-Arrangement

flat. Note that not all arrangements can be drawn as matroids in this way, or rather, they cannot be drawn and interpreted visually if higher ranks are involved. The matroid for the braid arrangement in 3 dimensions is shown in Figure 5.

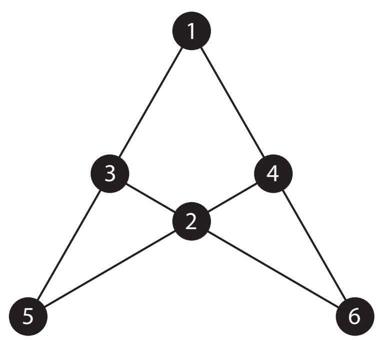

Figure 5: 2-Dimensional matroid representing the Braid 3-Arrangement

5 Irreducible Flats and Nested Sets

In this section we introduce the properties of reducibility and incomparability, and then the concept of nested sets, which will become useful later in the construction of neighborly partitions

5.1 Irreducibility and Incomparability

Definition 5.1. A central arrangement $\mathcal{A}$ is *irreducible* if $\mathcal{A}$ cannot be expressed as a product of two nonempty arrangements.

We define a product of arrangements as follows:

Definition 5.2. Let $\mathcal{A}_1$ be an arrangement in V_1 and $\mathcal{A}_2$ an arrangement in V_2, both nonempty. The **product** of $\mathcal{A}_1$ and $\mathcal{A}_2$ is

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H \times V_2 \mid H \in \mathcal{A}_1\} \cup \{ V_1 \times K \mid K \in \mathcal{A}_1\}$$

in $V_1 \times V_2$.

In the context of matroids, rank-1 irreducible flats are the hyperplanes themselves, rank-2 flats are sets of collinear points, and rank-3 flats are sets of coplanar points.

Definition 5.3. Two flats are **incomparable** if one is not contained in the other (thus 12 and 34 are incomparable, but 12 and 125 are not). A set of flats S is **pairwise incomparable** if every pair of flats in S are incomparable.

5.2 Nested Sets

Now with the concepts of reducibility and incomparability defined, we can describe what we call a *nested set*.

Definition 5.4. A subset S of the irreducible flats of $\mathcal{A}$ is a **nested set** if any pairwise incomparable subset $S' \subset S$, has a reducible join in $\mathcal{A}$, where the **join** is defined as the closure of the union of the elements in S'. Thus if the hyperplanes H_1, H_2, H_5 intersect to form one flat, then the closure of 12 is $\overline{12} = 125$.

We are generally only concerned with the *maximal* nested sets, since any subset of a nested set is itself nested. See the appendix for Mathematica packages that generate lists of irreducible flats and nested sets.

Example 5.1 (Pyramid)**.** The pyramid shown in Figure 6 is a rank-4 (3-dimensional) matroid representing an arrangement of five hyperplanes. The irreducible sets are $\{1, 2, 3, 4, 1234\}$. The output from nestedsets.m (see appendix) showing the nested sets for the pyramid arrangement is shown in Figure 7.

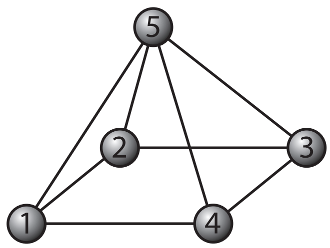

Figure 6: Matroid for the pyramid

```
{1}  {2}  {5}
{1}  {2}  {1, 2, 3, 4}
{1}  {3}  {5}
{1}  {3}  {1, 2, 3, 4}
{1}  {4}  {5}
{1}  {4}  {1, 2, 3, 4}
{2}  {3}  {5}
{2}  {3}  {1, 2, 3, 4}
{2}  {4}  {5}
{2}  {4}  {1, 2, 3, 4}
{3}  {4}  {5}
{3}  {4}  {1, 2, 3, 4}
```

Figure 7: Output of nestedsets.m run on the pyramid arrangement

Another example of a matroid, one that is frequently studied because of its simple yet still interesting structure, is the prism arrangement. Note that the prism is an example of a *3-generic* arrangement (see Definition 2.12).

Example 5.2 (Prism)**.** The prism (Figure 8) is comprised of 5 faces, two triangular and three quadrilateral. The matroid in Figure 8 is drawn in such a way that the "diagonal planes" ($\{1,4,6\}$, $\{2,3,5\}$, $\{3,2,6\}$, $\{4,1,5\}$, $\{5,2,4\}$ and $\{6,1,3\}$) are in fact coplanar, however diagonals such as 25 are not drawn, but are implicit for planes that contain four or more points. We make this convention to simplify the appearance of the matroid. Here the nested sets are of three different forms. The two triangular faces correspond to the nested sets $\{1,3,5\}$ and $\{2,4,6\}$. The second form of nested set contains any two vertices from a triangular face, along with another point such that their closure does not include any other points (this is essentially the same as the first type of nested set). These are sets of the form $\{1,4,6\}$, for example. The last type of nested sets in this example are those of the form $\{1,2,1234\}$, which contain two points whose closure forms a quadrilateral face.

There are many more complex matroids of higher rank that are not re-

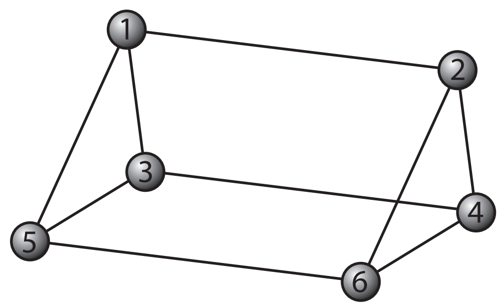

Figure 8: Matroid for the triangular prism

alizable with a matroid drawing in three or less dimensions. The geometric intuition from rank-3 and rank-4 arrangements helps in understanding the concept of nested sets, but the composition of nested sets is easily extended to higher rank arrangements.

6 Neighborly Partitions

The process of creating neighborly partitions is the final concept we must introduce to be able to determine the resonance varieties of a hyperplane arrangement. For the final two sections, we will follow along with the example of the Braid 3-arrangement (Figure 9).

6.1 Creating Partitions

We define a a vector λ_1 by assigning weights to each hyperplane, and defining *partitions* $\{A, B, C \ldots\}$ such that the followings rules hold true. For every intersection of hyperplanes, either all the planes intersecting are in the same partition, or it's not the case that all but one are in the same partition. A valid labeling should have a maximal number of partitions. In the example of the Braid 3-arrangement, since hyperplanes H_1 and H_2 intersect at infinity (which is the case with parallel lines in a projective diagram), we put them in the same partition in order to satisfy the rule of not having all but one intersecting hyperplane in the same partition. In this way, each pair of parallel lines, along with hyperplanes H_5 and H_6, are in the same partition. If we give them each a different label, then the four points where three

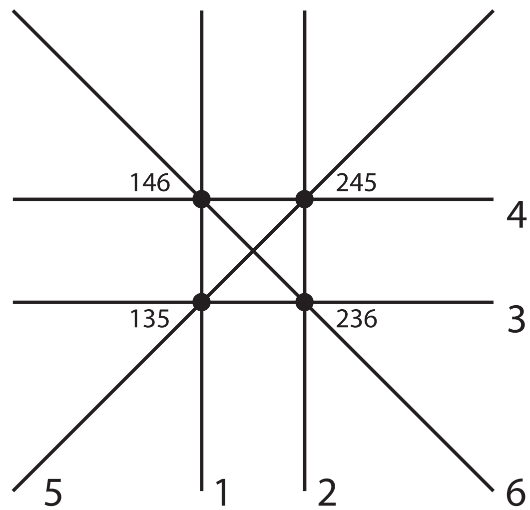

Figure 9: The Braid 3-arrangement example

hyperplanes meet will have hyperplanes from A, B, and C, satisfying the conditions.

6.2 Hyperplane Weights

Next we assign weights to each hyperplane to create linearly independent vectors $\lambda_1, \lambda_2, \ldots$. First choose a partition to set to zero for now, say B. Now assign a weight of one to each hyperplane in A. Now it is a game of adjusting nonzero weights for hyperplanes in A and giving appropriate weights to hyperplanes in other partitions, namely C, such that for each intersection including hyperplanes from different partitions, the sum of the hyperplane weights is equal to zero. In this manner, we obtain weights of $\{1, 1, 0, 0, -1, -1\}$. This is our vector λ_1. We can do the same process again, this time setting the weights in A to zero and the weights in B to one. In this way we obtain a maximum of two linearly independent vectors, and we define the matrix Λ as follows:

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

To better illustrate the process of partitioning, we give a more complex example below.

Example 6.1. We will divide the arrangement illustrated by the projective diagram in Figure 10. We begin by looking at the intersections of two hyperplanes. Any such pair of hyperplanes must belong to the same partition. Recall that parallel lines intersect at infinity, along with hyperplane 7. This results in the partition $\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$. We check the intersections to make sure the conditions are satisfied. In this case example none of the partitions need to be combined, as each intersection point has at least one hyperplane from each of three partitions.

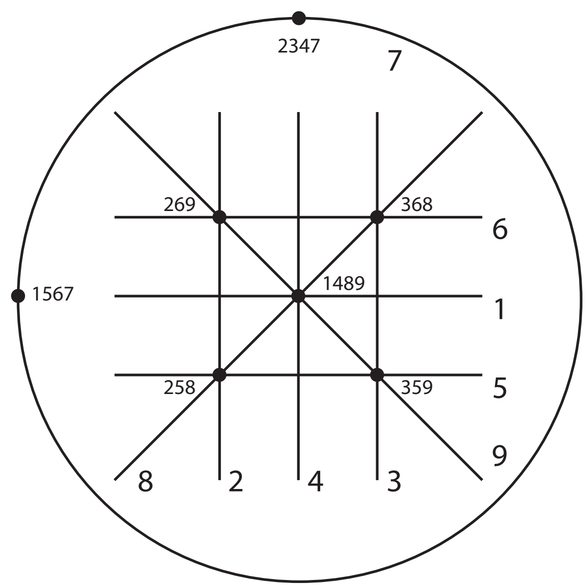

Figure 10: An example of a more complex partition

Now we must assign appropriate weights to create linearly independent vectors. First set the hyperplanes in B to zero. Now we see that some points of intersection contain two hyperplanes from A and one from C, while some contain one from A and two from C. The weights work out to zero if we assign the hyperplanes in A the values $\{2,1,1\}$, and the values $\{-2,-1,-1\}$ to C. We do a similar process after setting the weights in A to zero to get two linearly independent vectors and find a rank 2 Λ matrix.

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 2 & 1 & 1 & -2 & -1 & -1 \end{bmatrix}.$$

6.3 Generating the O-S Algebra

Now that we have our rank 2 matrix for the Braid 3-arrangement,

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix},$$

we can define elements in the ideal $I(\mathcal{A})$ that generates the Orlik-Solomon algebra $A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$. Recall that if a set of hyperplanes is dependent in the arrangement, then the boundary operator sends the product of the corresponding elements in the exterior algebra to zero. For example, the intersection 235 in the Braid 3-arrangement corresponds to the product $e_2e_3e_5$, written as e_{235} for convenience. The boundary operator ∂ sends this product to zero. By the definition of the boundary mapping,

$$\partial e_{235} = e_{23} - e_{25} + e_{35} \in I(\mathcal{A}).$$

Observe that we have two elements in the Orlik-Solomon algebra, $a_{\lambda_1} = a_1 + a_2 - a_5 - a_6$ and $a_{\lambda_2} = a_3 + a_4 - a_5 - a_6$. We see that $a_{\lambda_1} \wedge a_{\lambda_2} = 0$:

$$\begin{aligned} a_{\lambda_1} \wedge a_{\lambda_2} &= (a_1 + a_2) - (a_5 + a_6)] \wedge [(a_3 + a_4) - (a_5 + a_6) \\ &= (a_1 + a_2) \wedge (a_3 + a_4) - (a_1 + a_2) \wedge (a_5 + a_6) \\ &\qquad -(a_5 + a_6) \wedge (a_3 + a_4) + (a_5 + a_6) \wedge (a_5 + a_6) \\ &= a_{13} + a_{14} + a_{23} + a_{24} - a_{15} - a_{16} - a_{25} - a_{26} \\ &\qquad -a_{53} - a_{63} - a_{54} - a_{64} + 0 \\ &= a_{13} + a_{14} + a_{23} + a_{24} - a_{15} - a_{16} - a_{25} - a_{26} \\ &\qquad +a_{35} + a_{36} + a_{45} + a_{46} \\ &= (a_{13} - a_{16} + a_{26}) + (a_{14} - a_{15} + a_{45}) + (a_{23} - a_{25} + a_{35}) \\ &\qquad +(a_{24} - a_{26} + a_{46}) \\ &= a_{136} + a_{145} + a_{235} + a_{246} \\ &= 0 + 0 + 0 + 0 = 0 \end{aligned}$$

6.4 Theorems on Λ

Our Λ matrix for the Braid 3-arrangement can illustrate the application of the following theorem.

Theorem 6.1. *Given an irreducible arrangement $\mathcal{A}$, there exists a matrix Λ of rank 2 where $a_{\lambda_1} \wedge a_{\lambda_2} = 0$ iff for any intersection (flat) X of rank 2, either one of the following is true:*

(1) $|X| \geq 3$ *and* $\sum_{j \in X} \lambda_{i,j} = 0$ *for* $i = 1, 2$

(2) $\lambda_1(X)$ *is parallel to* $\lambda_2(X)$*, i.e., rank* $\begin{bmatrix} \lambda_{1j} \\ \lambda_{2j} \end{bmatrix} \leq 1$

Example 6.2. Let $X = 235$. Then according to case 1, $\lambda_{1,235}$ and $\lambda_{2,235}$ equal zero.

$$\begin{aligned} \lambda_{1,235} &= \lambda_{12} + \lambda_{13} + \lambda_{15} = 1 + 0 + -1 = 0, \\ \lambda_{2,235} &= \lambda_{22} + \lambda_{23} + \lambda_{25} = 0 + 1 + -1 = 0. \end{aligned}$$

Example 6.3. Let $X = 34$. Then $\lambda_1(34) = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $\lambda_2(34) = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Then according to case 2,

$$\text{rank} \begin{bmatrix} \lambda_1(34) \\ \lambda_2(34) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \leq 1.$$

Theorem 6.1 is a very special case of the following more powerful theorem:

Theorem 6.2. $a_{\lambda_1} \wedge a_{\lambda_2} \wedge \ldots \wedge a_{\lambda_p} = 0$ *iff for any maximal nested set* $\{X_1, \ldots, X_q\}$,

$$\textit{rank} \begin{bmatrix} \lambda_{1,X_1} & \lambda_{1,X_2} & \ldots & \lambda_{1,X_q} \\ \lambda_{2,X_1} & \lambda_{2,X_2} & \ldots & \lambda_{2,X_q} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{p,X_1} & \lambda_{p,X_2} & \ldots & \lambda_{p,X_q} \end{bmatrix} < p.$$

The proof, however, involves tropical geometry, and is beyond the scope of this research project. For the 3-dimensional version of the above theorem, we have the following:

Theorem 6.3. *$a_{\lambda_1} \wedge a_{\lambda_2} \wedge a_{\lambda_3} = 0$ iff for any nested set $\{X_1, \ldots, X_p\}$, the matrix*

$$\Lambda_{(X_1,\ldots,X_p)} = \begin{bmatrix} \lambda_{1,X_1} & \ldots & \lambda_{1,X_p} \\ \lambda_{2,X_1} & \ldots & \lambda_{2,X_p} \\ \lambda_{3,X_1} & \ldots & \lambda_{3,X_p} \end{bmatrix}$$

has rank ≤ 2.

Now we have the tools necessary to define a neighborly partition.

6.5 Neighborly Partitions and Hypergraphs

Definition 6.1. When $p = 2$, define a *neighborly partition* by a graph with hyperplanes as vertices and edges (ij) iff rank $\begin{bmatrix} \lambda_{1i} & \lambda_{1j} \\ \lambda_{2i} & \lambda_{2j} \end{bmatrix} < 2$.

The same idea extends to the case of $p = 3$ (and further) to form *neighborly hypergraphs*.

Definition 6.2. When $p = 3$, define a *neighborly hypergraph* by a graph with hyperplanes as vertices and edges (ijk) iff rank $\begin{bmatrix} \lambda_{1i} & \lambda_{1j} & \lambda_{1k} \\ \lambda_{2i} & \lambda_{2j} & \lambda_{2k} \\ \lambda_{3i} & \lambda_{3j} & \lambda_{3k} \end{bmatrix} < 3$.

6.6 Polychrome and Monochrome Flats

In this section we look at the properties of a neighborly hypergraph more closely. Define a hypergraph Γ on $[n]$ whose edges are triples $\{i, j, k\}$ with

$$\text{rank} \begin{bmatrix} \lambda_{1i} & \lambda_{1j} & \lambda_{1k} \\ \lambda_{2i} & \lambda_{2j} & \lambda_{2k} \\ \lambda_{3i} & \lambda_{3j} & \lambda_{3k} \end{bmatrix} \leq 2.$$

Now consider the following lemma:

Lemma 6.4. *Let X be an irreducible flat with $\lambda_{i,X} \neq 0$ for some $i = 1, 2, 3$. Then X is a clique of Γ, that is, every triple $\{i, j, k\}$ in X is an edge of Γ.*

Proof. We show that the matrix

$$\left[\ \underline{\lambda_i} \mid \underline{\lambda_j} \mid \underline{\lambda_k} \mid \underline{\lambda_X}\ \right] = \begin{bmatrix} \lambda_{1i} & \lambda_{1j} & \lambda_{1k} & \lambda_{1,X} \\ \lambda_{2i} & \lambda_{2j} & \lambda_{2k} & \lambda_{1,X} \\ \lambda_{3i} & \lambda_{3j} & \lambda_{3k} & \lambda_{1,X} \end{bmatrix} = \begin{bmatrix} \lambda_1(X) \\ \lambda_2(X) \\ \lambda_3(X) \end{bmatrix}$$

has rank ≤ 2. We know that the rank of $\left[\ \underline{\lambda_i} \mid \underline{\lambda_j} \mid \underline{\lambda_X}\ \right]$ is ≤ 2 and all three columns are nonzero, similarly for i, k and j, k. Then $\underline{\lambda_j}$ is in the span of $(\underline{\lambda_i}, \underline{\lambda_X})$, and the same is true for $\underline{\lambda_k}$. Thus the column space is spanned by $(\underline{\lambda_i}, \underline{\lambda_X})$, and so the rank of matrix is ≤ 2.

It then follows that $\{i, j, k\}$ is an edge of Γ, since the determinant of $\left[\ \underline{\lambda_i} \mid \underline{\lambda_j} \mid \underline{\lambda_k}\ \right]$ is zero. □

Definition 6.3. We say the X is **monochrome** if $\begin{bmatrix} \lambda_1(X) \\ \lambda_2(X) \\ \lambda_3(X) \end{bmatrix}$ has rank two.

The name *monochrome* comes from the notion of the hyperplanes in X forming a clique in the neighborly hypergraph, as every triple in X is in the hypergraph.

Definition 6.4. We call X **polychrome** if $\lambda_1(X) = \lambda_2(X) = \lambda_3 X = 0$

We call such a flat *polychrome* since each of the hyperplanes in X are disconnected in the hypergraph.

6.7 Transitivity in Neighborly Partitions

We conclude this section with a look at the transitive properties of neighborly partitions. This 'transitivity' property extends from the idea of a transitive graph, a graph where for any three vertices (i, j, k) in which i and j are connected and j and k are connected, i and k are connected as well.

Theorem 6.5. *Let Γ be the neighborly partition of an arrangement $\mathcal{A}$. If X is a flat of rank 2 and $i \in X$ with $X - \{i\}$ a clique of Γ, then X is a clique of Γ.*

Proof. Suppose X is not a clique in Γ. Then $\lambda_{1,X} = 0 = \lambda_{2,X}$. Since X is not a clique, $\exists\ j \neq i$ where $i, j \in X$ and

$$\operatorname{rank} \begin{bmatrix} \lambda_{1i} & \lambda_{1j} \\ \lambda_{2i} & \lambda_{2j} \end{bmatrix} < 2.$$

since the two column vectors are parallel But since $\lambda_{1,X} = \lambda_{2,X} = 0$, it follows that

$$\det \begin{bmatrix} \lambda_{1,X} & \lambda_{1j} \\ \lambda_{2,X} & \lambda_{2j} \end{bmatrix} = 0.$$

Therefore,

$$\begin{aligned} 0 &= \det \begin{bmatrix} \lambda_{1,X} & \lambda_{1j} \\ \lambda_{2,X} & \lambda_{2j} \end{bmatrix} \\ &= \det \begin{bmatrix} \sum_k \lambda_{1k} & \lambda_{1j} \\ \sum_k \lambda_{2k} & \lambda_{2j} \end{bmatrix} \\ &= \sum_{k \in X} \det \begin{bmatrix} \lambda_{1k} & \lambda_{1j} \\ \lambda_{2k} & \lambda_{2j} \end{bmatrix}. \end{aligned}$$

This last expression simplifies to $\det \begin{bmatrix} \lambda_{1i} & \lambda_{1j} \\ \lambda_{2i} & \lambda_{2j} \end{bmatrix}$, however, since every j, k in $X - \{i\}$ are parallel by the hypothesis. But we have already stated that this determinant does not equal zero. Thus our assumption was false and X is a clique. □

Part of our research included trying to generalize the condition of transitivity to hypergraphs.

7 Resonance Varieties

7.1 The p-th Cohomology H^p

Let A be the O-S algebra. Let $\mathcal{A}$ be an arrangement where $\text{rank}(\mathcal{A}) = l$ and $n = |\mathcal{A}|$. We have the following cochain complex:

$$0 \to A^0 \xrightarrow{a} A^1 \xrightarrow{a} \cdots \to A^p \to \cdots \to A^l \to 0.$$

Here $A^0 \cong \mathbb{C}$ and $A^1 \cong \bigoplus_{i=1}^n \mathbb{C}e_i \cong \mathbb{C}^n$. A^p is spanned by $e_{i_1} \wedge \cdots \wedge e_{i_p}$, and $a \cdot (e_{i_1} \wedge \cdots \wedge e_{i_p}) = (\sum \lambda_i e_i)(e_{i_1} \wedge \cdots \wedge e_{i_p}) \in A^{p+1}$. Observe that

$$A^{p-1} \xrightarrow{0} A^{p+1},$$

and so $\text{im}(a) \subseteq \ker(a)$.

We define the p-th cohomology of (A, a) as

$$H^p(A, a) = \frac{\ker(a : A^p \longrightarrow A^{p+1})}{\text{im}(a : A^{p-1} \longrightarrow A^p)}.$$

It can be shown that for generic a, $H^p(A, a) = 0 \ \forall p$.

7.2 The p-th Resonance Variety

The p-th resonance variety is

$$\mathcal{R}^p = \{a \in A^1 \cong \mathbb{C}^n \mid H^p(A, a) \neq 0\}.$$

This paper only concerns itself with the first degree resonance variety ($p = 1$), which, put more simply, is

$$\mathcal{R}^1 = \{a_1 \in A^1 \mid \exists\ a_2 \in A^1, a_1 a_2 = 0, a_2 \text{ not a scalar multiple of } a_1\}.$$

Again, we look to the Braid 3-arrangement example to illustrate this structure. The degree-1 resonance variety of the Braid 3-arrangement consists of the union of five 2-dimensional linear subspaces of $\mathbb{C}$. For each rank 2 flat, we get a 2-dimensional subspace. For example, let $X = 136$. The subspace is spanned by $e_1 - e_3$ and $e_3 - e_6$. The fifth and final subspace is spanned by $e_1 + e_2 - e_5 - e_6$ and $e_3 + e_4 - e_5 - e_6$.

The resonance varieties are an important invariant of hyperplane arrangements. The *Tangent Cone Theorem*, for example, implies that they give some information about the fundamental group of the complement.

8 Conclusion

In the final weeks of the program, much of the effort was put into understanding neighborly partitions and resonance varieties, in addition to trying to find a condition on neighborly hypergraphs that was analogous to transitivity on neighborly partitions. Some of the research period was also spent concentrating on graphic arrangements, and then a brief amount of time was spent investigating *Tutte Polynomials* before focusing on neighborly partitions. One week was devoted in large part to learning how to program in Mathematica, which resulted in the four programs shown in the appendix. In overview, much of the summer was dedicated to a review of the literature, leading to some valuable experience in applying algebraic concepts to an area of research completely new to me, and presenting my findings in an hour-long presentation given in the closing week.

A Mathematica Packages Source Code

The first three Mathematica packages (`3generic.m`, `irreducibles.m`, and `nestedsets.m`) were written with the goal of being able to quickly compute the nested sets of an arrangement. Shown here are the programs after final efforts to enhance their runtime were made. Also shown is the program `nullspaces.m`, which arose from working on transitivity conditions for the neighborly hypergraph.

A.1 3-Generic Test

```
(* ::Package:: *)

(* Written by Andres Perez on 07/07/08.
3generictest.m
Test for 3-genericity. Input a matrix a of a rank 4 arrangement *)

(* The following code is taken from flats.m written by Michael Falk. It
takes a list of dependence equations for an arbitrary arrangement "a" and
creates "flats", a list of all flats, ordered by rank. *)

Module[{n,m,r,seq,KSets},
n=Length[a];
m=Length[a[[1]]];
r=m-Length[NullSpace[a]];
KSets[list_,k_]:=Block[{output={},counter={},
characteristic={}  },
Do[AppendTo[counter,1],{i,1,k}];
Do[AppendTo[counter,0],{i,1,Length[list]-k}];
characteristic=Permutations[counter];
Do[AppendTo[output,
  Complement[Union[list characteristic[[i]]],{0}]],
  {i,1,Length[characteristic]}];output];
seq[k_]:=KSets[Range[n],k];
flats=Range[r+1];
flats[[1]]={{}};
Do[q=k+1;flats[[q]]={};
Do[switch=0;Do[If[
```

```
Intersection[s=seq[k][[i]],flats[[q,j]]]==s,
Return[switch=switch+1]],
{j,1,Length[flats[[q]]]}];If[switch==0,
 If[Length[NullSpace[a[[seq[k][[i]]]]]]==m-k,
 f=seq[k][[i]];
Do[If[Length[NullSpace[a[[Union[Append[seq[k][[i]],l]]]]]]==m-k,
f=Union[Append[f,l]]],
{l,1,n}];flats[[q]]=Sort[Union[Append[flats[[q]],f]]]]],
{i,1,Binomial[n,k]}
],
{k,1,r-1}];

flats[[r+1]]={Range[n]};
]

switch=0;
Do[
If[Length[flats[[3,i]]]!=2,switch=1],{i,Length[flats[[3]]]}]
Print["The set of flats for the arrangement given by "];
Print[Panel[TableForm[a], BaseStyle -> {"StandardForm", Larger}]];
If[switch!=0,
Print["is not 3-generic."],
Print["is 3-generic."]]
```

A.2 Irreducibles List

```
(* ::Package:: *)

(* Written by Andres Perez on 07/07/08.
irreducibles.m
Returns the set of irreducible flats or rank 1, 2 and 3, listed
according to rank for an arbitrary arrangement *)

(* The following code is taken from flats.m written by Michael Falk. It
takes a list of dependence equations for an arbitrary arrangement "a" and
creates "flats", a list of all flats, ordered by rank. *)

Module[{n,m,r,seq,KSets},
```

```
n=Length[a];
m=Length[a[[1]]];
r=m-Length[NullSpace[a]];
KSets[list_,k_]:=Block[{output={},counter={},
characteristic={}  },
Do[AppendTo[counter,1],{i,1,k}];
Do[AppendTo[counter,0],{i,1,Length[list]-k}];
characteristic=Permutations[counter];
Do[AppendTo[output,
  Complement[Union[list characteristic[[i]]],{0}]],
  {i,1,Length[characteristic]}];output];
seq[k_]:=KSets[Range[n],k];
flats=Range[r+1];
flats[[1]]={{}};
Do[q=k+1;flats[[q]]={};
Do[switch=0;Do[If[
Intersection[s=seq[k][[i]],flats[[q,j]]]==s,
Return[switch=switch+1]],
{j,1,Length[flats[[q]]]}];If[switch==0,
 If[Length[NullSpace[a[[seq[k][[i]]]]]]==m-k,
 f=seq[k][[i]];
Do[If[Length[NullSpace[a[[Union[Append[seq[k][[i]],l]]]]]]==m-k,
f=Union[Append[f,l]]],
{l,1,n}];flats[[q]]=Sort[Union[Append[flats[[q]],f]]]]]],
{i,1,Binomial[n,k]}
],
{k,1,r-1}];

flats[[r+1]]={Range[n]};
]

(* The remaining code takes "flats" and creates "irreducibles", a list of
irreducible flats of ranks 1, 2 and 3. *)

(* Initialize the set of flats for each rank. The singleton sets comprise
the irreducible flats of rank 1 *)
irred1=flats[[2]];
irred2=flats[[3]];
```

```
irred3=flats[[4]];

(* In the set of rank 2 flats, if the i-th flat with the j-th entry
removed is in the set of rank 1 flats, remove the i-th flat from
"irred2". Repeat for all "j" from 1 to the size of the flat, for all "i"
from 1 to the size of the list of rank 2 flats. *)

Do[
If[MemberQ[flats[[2]],Drop[flats[[3,i]],{j}]],
irred2=Complement[irred2,{flats[[3,i]]}]],
{i,Length[flats[[3]]]},
{j,Length[flats[[3,i]]]}
]

(* In the set of rank 3 flats, if the i-th flat with the j-th entry
removed is in the set of rank 2 flats, remove the i-th flat from
"irred3". Repeat for all "j" from 1 to the size of the flat, for all "i"
from 1 to the size of the list of rank 3 flats. *)

Do[
If[MemberQ[flats[[3]],Drop[flats[[4,i]],{j}]],
irred3=Complement[irred3,{flats[[4,i]]}]],
{i,Length[flats[[4]]]},
{j,Length[flats[[4,i]]]}
]

(* Print the output: A table of the dependence equations, along with the
set of irreducible flats of rank 1, 2 and 3, in that order. *)

Print["The list of irreducible flats of rank 1, 2 and 3
for the arrangement"];
Print[Panel[TableForm[a], BaseStyle -> {"StandardForm", Larger}]];
Print["are as follows"];
Print["Rank 1:"];
Print[Panel[irred1, BaseStyle -> {"StandardForm", Larger}]];
Print["Rank 2:"];
Print[Panel[irred2, BaseStyle -> {"StandardForm", Larger}]];
Print["Rank 3:"];
```

```
Print[Panel[irred3, BaseStyle -> {"StandardForm", Larger}]];
```

A.3 Nested Sets

```
(* ::Package:: *)

(* Written by Andres Perez on 07/09/08.
nestedsets.m
Returns the nested sets of an arrangement *)

(* The following code is taken from flats.m written by Michael Falk. It
takes a list of dependence equations for an arbitrary arrangement "a" and
creates "flats", a list of all flats, ordered by rank. *)

Module[{n,m,r,seq,KSets},
n=Length[a];
m=Length[a[[1]]];
r=m-Length[NullSpace[a]];
KSets[list_,k_]:=Block[{output={},counter={},
characteristic={}  },
Do[AppendTo[counter,1],{i,1,k}];
Do[AppendTo[counter,0],{i,1,Length[list]-k}];
characteristic=Permutations[counter];
Do[AppendTo[output,
  Complement[Union[list characteristic[[i]]],{0}]],
  {i,1,Length[characteristic]}];output];
seq[k_]:=KSets[Range[n],k];
flats=Range[r+1];
flats[[1]]={{}};
Do[q=k+1;flats[[q]]={};
Do[switch=0;Do[If[
Intersection[s=seq[k][[i]],flats[[q,j]]]==s,
Return[switch=switch+1]],
{j,1,Length[flats[[q]]]}];If[switch==0,
 If[Length[NullSpace[a[[seq[k][[i]]]]]]==m-k,
 f=seq[k][[i]];
Do[If[Length[NullSpace[a[[Union[Append[seq[k][[i]],l]]]]]]==m-k,
f=Union[Append[f,l]]],
```

```
{l,1,n}];flats[[q]]=Sort[Union[Append[flats[[q]],f]]]]],
{i,1,Binomial[n,k]}
],
{k,1,r-1}];

flats[[r+1]]={Range[n]};
]

(* The following code is taken from irreducibles.m by Andres Perez. It
takes "flats" and creates lists of irreducible flats of ranks 1, 2 and 3,
respectively. *)

irred1=flats[[2]];
irred2=flats[[3]];
irred3=flats[[4]];
Do[
If[MemberQ[flats[[2]],Drop[flats[[3,i]],{j}]],
irred2=Complement[irred2,{flats[[3,i]]}]],
{i,Length[flats[[3]]]},
{j,Length[flats[[3,i]]]}
]
Do[
If[MemberQ[flats[[3]],Drop[flats[[4,i]],{j}]],
irred3=Complement[irred3,{flats[[4,i]]}]],
{i,Length[flats[[4]]]},
{j,Length[flats[[4,i]]]}
]

(* The remaining code returns a list of the nested sets for the
arrangement. *)

center=Union[Flatten[flats]];
allflats=Drop[Flatten[flats,1],1];
irred=Union[irred1,irred2,irred3,{center}];

(* latticejoin also can take a set of flats and determines the closure of
their collective join. *)
```

```
latticejoin[set_,flatslist_]:=Block[{minimal=Union[Flatten[set]],
ljoin=center},
Do[
If[Intersection[flatslist[[k]],minimal]==minimal &&
Intersection[flatslist[[k]],ljoin]==flatslist[[k]],
Return[ljoin=flatslist[[k]]]
],
{k,Length[flatslist]}
];
ljoin
];

(* subsetQ returns true if the given set is a subset of any of the
elements in a given list. *)

subsetQ[list_,set_]:=Block[{output=False},
Do[
If[Intersection[set,list[[i]]]==set,
Return[output=True]
],
{i,Length[list]}
];
output
]

(* pwincomp returns true if the given subset is pairwise-incomparable. *)

pwincomp[subset_]:=Block[{output=True},
If[Length[subset]<2,
Return[output=False],
Do[
If[Intersection[subset[[i]],subset[[j]]]==subset[[i]] && i!=j,
Return[output=False]; Break[]],
{i,Length[subset]},
{j,Length[subset]}
];
];
output
```

```
]

(* nestQ determines whether a given set of subsets is a nested set. *)

nestedQ[set_,flatslist_,irred_]:=Block[{output=True},
subsets=Drop[Subsets[set],1];
Do[
If[pwincomp[t] && MemberQ[irred,latticejoin[t,flatslist]],
Return[output=False];Break[]
],
{t,subsets}
];
output
]

(* Initialize and build up the list of maximal nested sets. *)

nested={};
biglist=Reverse[Drop[Drop[Subsets[irred],-1],1]];
add=False;
Do[
If[subsetQ[nested,s]==False && nestedQ[s,allflats,irred],
Return[add=True]
]
If[add,AppendTo[nested,s];Return[add=False]],
{s,biglist}
]

(* Clip off the element [n] from each member of the list "nested" (We can
assume that it is a member of every nested set), and re-order the list in
increasing order. *)

Do[nested[[i]]=Complement[nested[[i]],{center}],
{i,Length[nested]}
]
nested=Reverse[nested];

(* Print the list of nested sets *)
```

```
Print[Panel[Grid[a],
BaseStyle -> {"StandardForm", Larger}]];
Print["The (maximal) nested sets for the arrangement given above are:"];
Print[Panel[Grid[nested, Alignment -> Left],
BaseStyle -> {"StandardForm", Larger}]];
```

A.4 NullSpaces

```
(* ::Package:: *)

(* Written by Andres Perez on 07/09/08.
nullspaces.m *)

(* The following code is taken from flats.m written by Michael Falk. It
takes a list of dependence equations for an arbitrary arrangement "a" and
creates "flats", a list of all flats, ordered by rank. *)

Module[{n,m,r,seq,KSets},
n=Length[a];
m=Length[a[[1]]];
r=m-Length[NullSpace[a]];
KSets[list_,k_]:=Block[{output={},counter={},
characteristic={}  },
Do[AppendTo[counter,1],{i,1,k}];
Do[AppendTo[counter,0],{i,1,Length[list]-k}];
characteristic=Permutations[counter];
Do[AppendTo[output,
  Complement[Union[list characteristic[[i]]],{0}]],
  {i,1,Length[characteristic]}];output];
seq[k_]:=KSets[Range[n],k];
flats=Range[r+1];
flats[[1]]={{}};
Do[q=k+1;flats[[q]]={};
Do[switch=0;Do[If[
Intersection[s=seq[k][[i]],flats[[q,j]]]==s,
Return[switch=switch+1]],
{j,1,Length[flats[[q]]]}];If[switch==0,
```

```
 If[Length[NullSpace[a[[seq[k][[i]]]]]]==m-k,
 f=seq[k][[i]];
Do[If[Length[NullSpace[a[[Union[Append[seq[k][[i]],l]]]]]]==m-k,
f=Union[Append[f,l]]],
{l,1,n}];flats[[q]]=Sort[Union[Append[flats[[q]],f]]]]],
{i,1,Binomial[n,k]}
],
{k,1,r-1}];

flats[[r+1]]={Range[n]};
]

(* The following code is taken from irreducibles.m by Andres Perez. It
takes "flats" and creates "irreducibles", a list of irreducible flats of
ranks 1, 2 and 3. *)

irred1=flats[[2]];
irred2=flats[[3]];
irred3=flats[[4]];
Do[
If[MemberQ[flats[[2]],Drop[flats[[3,i]],{j}]],
irred2=Complement[irred2,{flats[[3,i]]}]],
{i,Length[flats[[3]]]},
{j,Length[flats[[3,i]]]}
]
Do[
If[MemberQ[flats[[3]],Drop[flats[[4,i]],{j}]],
irred3=Complement[irred3,{flats[[4,i]]}]],
{i,Length[flats[[4]]]},
{j,Length[flats[[4,i]]]}
]

(* The remaining code creates an incidence matrix of the rank 3
irreducible flats. Then it calculates the NullSpace of the matrix and its
submatrices. When the dimension of the nullspace is 3 or greater, it is
printed out. *)

(* Create a matrix of rank 3 irreducible flats *)
```

```
Print["The matrix of rank 3 irreducible flats:"];
irrmatrix=Table[If[MemberQ[irred3[[i]],j],1,0],
{i,Length[irred3]},{j,Length[flats[[2]]]}];
Print[Panel[Grid[irrmatrix], BaseStyle -> {"StandardForm", Larger}]];

(* Prints out the NullSpace of the matrix *)

Print["has NullSpace"]
Print[Panel[NullSpace[irrmatrix],
BaseStyle -> {"StandardForm", Larger}]];

(* Prints out the NullSpace of a submatrix if the dimension is greater
than or equal to 3. Otherwise, prints out a statement telling the user
that none were found. *)

none=True;
Do[
sub=Drop[irrmatrix,{i,i}];
If[Length[sub]!=0 && Length[NullSpace[sub]]>=3,
Print["------------------"]
Print["The NullSpace with the row"]
Print[Panel[Part[irrmatrix,i],
BaseStyle -> {"StandardForm", Larger}]]
Print["removed is"]
Print[Panel[NullSpace[sub],
BaseStyle -> {"StandardForm", Larger}]]
Return[none=False];
],
{i,Length[irred3]}
]

If[none,Print["There are no submatrices with NullSpace dimension
\[GreaterEqual]3."]]
```

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