A note on a superlinear elliptic Dirichlet problem

John M. Neuberger and Antonio R. Vargas

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Abstract

We show that a special subset of solution candidates (called S_1) for a superlinear elliptic Dirichlet problem is path-connected. It was shown by Castro et al. in [2] that the element of this set that minimizes the action functional is a solution to the PDE and changes sign exactly once.

1 Introduction

In this section we introduce the context of the problem and define relevant formulae and sets. In Section 2 we state several lemmas required for the proof of the result. Lastly, we prove the path-connectedness of the set S_1 in Section 3.

We are concerned with the boundary value problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{1}$$

where Δ is the Laplacian operator, Ω is a smooth subset of $\mathbb{R}^n$, and f has the following restrictions:

$$\begin{aligned} &\text{(a)} \quad f \in C^1(\mathbb{R}, \mathbb{R}), \\ &\text{(b)} \quad f'(u) > \frac{f(u)}{u} \text{ for } u \neq 0, \\ &\text{(c)} \quad \lim_{|u|\to\infty} \frac{f(u)}{u} = \infty. \end{aligned}$$

We also assume that if $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ are the eigenvalues of $-\Delta$ then $f'(0) < \lambda_1$.

Let H be the Sobolev space $H_0^{1,2}(\Omega)$ with inner product

$$\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \; dx$$

(see [1]). We define the action functional $J : H \to \mathbb{R}$ by

$$J(u) = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 - F(u) \right\} dx \tag{2}$$

where $F(u) = \int_0^u f(t)dt$. Next we calculate the directional derivative of J

$$J'(u)(v) = \langle \nabla J(u), v \rangle = \int_\Omega (\nabla u \cdot \nabla v - f(u)v)dx \tag{3}$$

for all $v \in H$. Using integration by parts on (3) one can see that u is a weak (and, consequently, a classical) solution to (1) if and only if it is a critical point of (2). Finally we define $\gamma(u) = J'(u)(u)$, $u_+(x) = \max\{u(x), 0\}$, and $u_-(x) = \min\{u(x), 0\}$ along with the sets

$$S = \{u \in H - \{0\} \mid \gamma(u) = 0\},$$

$$\hat{S} = \{u \in S \mid u_+, u_- \neq 0\}, \quad G^+ = \{u \in S \mid u > 0\},$$

$$S_1 = \{u \in \hat{S} \mid u_+, u_- \in S\}, \quad G^- = \{u \in S \mid u < 0\}.$$

Note that if u is a solution to (1) not identically zero then $u \in S$. Also, it should be stated that if u is in H then u_+ and u_- are also in H (see [3]).

It was proved in [2] that, under some additional restrictions on f, (1) has a positive solution, a negative solution, and a solution that changes sign exactly once. The sign changing solution is the element of S_1 that minimizes the action functional J.

2 Preliminary lemmas

Lemma 2.1. *The function $h : H \to H$ defined by $h(u) = u_+$ is continuous.*

Lemma 2.2. *If $u \in H - \{0\}$ then there exists a unique $\bar{\lambda} = \bar{\lambda}(u) \in (0, \infty)$ such that $\lambda u \in S$.*

Lemma 2.3. *The function $\bar{\lambda} \in C^1(S^\infty, (0, \infty))$. The set S is closed, unbounded, and a connected C^1-submanifold of h diffeomorphic to $S^\infty = \{u \in H \mid ||u|| = 1\}$.*

We refer the reader to [2] for proofs of these lemmas.

3 Proof of result

Theorem 3.1. *S_1 is path-connected.*

Proof. Let u and v be elements in S_1.

Case 1: There exist elements x and y in Ω such that $u(x), v(x) > 0$ and $u(y), v(y) < 0$. Take the convex linear combination

$$p(t) = (1 - t)u + tv.$$

Note that $p(t)$ is sign-changing (and consequently nonzero) for $t \in [0, 1]$. We claim that $p(t)_+$ and $p(t)_-$ project onto the paths

$$\{\alpha(t)p(t)_+ \in G^+ \mid t \in [0, 1]\}$$

and

$$\{\beta(t)p(t)_- \in G^- \mid t \in [0,1]\}$$

where $\alpha, \beta \in C([0,1], \mathbb{R}^+)$. This follows from the fact that p is continuous and from Lemmas 2.1 and 2.3. Define

$$q(t) = \alpha(t)p(t)_+ + \beta(t)p(t)_-.$$

Note that $\alpha(t)p(t)_+$ and $\beta(t)p(t)_-$ have disjoint support and so

$$q(t)_+ = \alpha(t)p(t)_+,$$
$$q(t)_- = \beta(t)p(t)_-.$$

Since $q(t)_+$ is in G^+ and $q(t)_-$ is in G^- for $t \in [0,1]$, $q(t)$ is in S_1 for $t \in [0,1]$. Further, $q(0) = u$ and $q(1) = v$ (from Lemma 2.2).

Case 2: There do not exist elements x and y in Ω such that $u(x), v(x) > 0$ and $u(y), v(y) < 0$. We aim to contruct a function $w \in S_1$ that satisfies the conditions of Case 1 for u and v. We would then construct a path as per above from u to w and then from w to v. That S_1 is path-connected would follow from such a construction.

First, take four distinct elements x_1, x_2, x_3, and x_4 in Ω such that $u(x_1) > 0$, $v(x_2) > 0$, $u(x_3) < 0$, and $v(x_4) < 0$. By the continuity of the elements of H we can construct four disjoint open sets A_1, A_2, A_3, and A_4 in Ω about the x_i such that:

$$(a) \quad \forall x \in A_1, u(x) > 0,$$
$$(b) \quad \forall x \in A_2, v(x) > 0,$$
$$(c) \quad \forall x \in A_3, u(x) < 0,$$
$$(d) \quad \forall x \in A_4, v(x) < 0.$$

Construct a function $w^* \in H$ that is positive on A_1 and A_2, negative on A_3 and A_4, and any sign elsewhere. By Lemma 2.2 there exist unique constants $c_1, c_2 \in (0, \infty)$ such that $c_1 w^*_+$ is in G^+ and $c_2 w^*_-$ is in G^-. Define

$$w = c_1 w^*_+ + c_2 w^*_-.$$

Since $c_1 w^*_+$ and $c_2 w^*_-$ have disjoint support, $w_+ = c_1 w^*_+$ and $w_- = c_2 w^*_-$. It follows that w is in S_1 and meets the requirements of Case 1 for u and v. □

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] A. Castro, J. Cossio, and J. M. Neuberger. A sign-changing solution for a superlinear Dirichlet problem. *Rocky Mountain Journal of Mathematics*, 27(4):1041–1053, 1997.

[3] D. Kinderlehrer and G. Stampacchia. *Introduction to Variational Inequalities and Their Applications.* Academic Press, New York, 1979.