What is a Number?

Marc A. Fabbri
Department of Mathematics
Arizona State University
Tempe, AZ 85287

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Abstract

It has been stated that the only satisfactory definition of a number is "an element of a field". It took Whitehead and Russel 150 pages to arrive at the definition of the number one and Three Dog Night characterized the number two as "the lonliest number since the number one."

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1 The Natural Numbers

The real reason why the elements of $\mathbb{N}$ are called 'natural numbers' was explained by Kronecker when he stated:

> Die ganzen hat der liebe Gott gemacht, alles andere ist Menschenwerk. [God made the whole numbers, all the rest is the work of Man.] (KRONECKER, Jahresber. DMV 2, S. 19)

Definition 1.1. The natural numbers form a set $\mathbb{N}$, containing a distinguished element 0, called zero, together with a successor function $S : \mathbb{N} \to \mathbb{N}$ which satisfies the following axioms:

1. S is injective
2. $0 \notin S(\mathbb{N})$
3. If a subset $M \subset \mathbb{N}$ contains 0 and is mapped into itself by S then $M = \mathbb{N}$.

The existence of the natural numbers depends on the *axiom of infinity*, [E$^+$91], and the uniqueness is gauranteed by the following result [Ded88].

Theorem 1.2 (Dedekind 1888). *Let A be an arbitrary set containing an element $a \in A$ and g a mapping $g : A \to A$. Then there is a one-to-one mapping $\phi : \mathbb{N} \to A$ satisfying $\phi(0) = a$ and making the following diagram commute:*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{S} & \mathbb{N} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ A & \xrightarrow[g]{} & A \end{array}$$

2 The Integers

Every integer can be expressed as the difference between two natural numbers. In fact we introduce the integers $\mathbb{Z}$ as the set of equivalence classes under the following equivalence relation on $\mathbb{N} \times \mathbb{N}$:

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad a + d = b + c \tag{1}$$

Remark 2.1. Every integer x can be expressed in the form

$$x = \begin{cases} 2n, & \text{if } x \text{ is } \textit{even} \\ 2n + 1, & \text{if } x \text{ is } \textit{odd} \end{cases} \tag{2}$$

for some integer n.

Theorem 2.2. *The product of two odd integers is an odd integer.*

Proof. Since x and y are two odd integers, they can be written $x = 2n + 1$ and $y = 2m + 1$ for some integers n and m. Thus

$$\begin{aligned} xy &= (2n + 1)(2m + 1) \\ &= 4nm + 2n + 2m + 1 \\ &= 2(2nm + n + m) + 1 \end{aligned}$$

and, in view of equation (2), we conclude xy is odd. □

product	*even*	*odd*
even	even	even
odd	even	odd

Table 1: Products of Integers

We leave it as an exercise for reader to prove that the product of an even integer with any integer is even and we summarize this in a Table 1.

3 Prime Numbers

A *prime number* is a natural number that is divisible only 1 and itself. Euclid proved that there are infinitely many primes. Since every natural number can be written as a product of powers of primes in exactly one way, we think of the primes as the 'building blocks' for the natural numbers.

Later Gauss proved his famous *Prime Number Theorem* which states that the number of primes $A(n)$ not exceeding n is asymptotically equivalent to $n/\log n$. In Figure 1 we plot the graph of this function over a large domain.

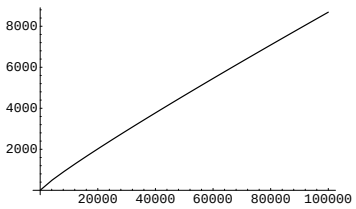

Figure 1: Distribution of Primes

References

[Ded88] R. Dedekind. *Was sind und was sollen die Zahlen?* Braunschweig, 1888.

[E+91] H.D. Ebbinghaus et al. *Numbers.* Springer-Verlag, 1991.

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